A Gravitomagnetic Extension of Newton-Cartan Theory

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Abstract

Essential to the framework of Newtonian theory are the concepts of absolute time, i.e. of a time that is the same for all observers, and of Euclidean spatial geometry. We represent an argument that the gravitational equivalence principle can be meaningfully understood from this Newtonian perspective and integrated into a theory of gravitation that regards the gravitational force as indistinguishable from the so-called inertial forces, with this equivalence mathematically described in a given frame of reference by an affine connection on the spacetime, as originally done by Élie Cartan. This model of gravitation serves as a near-field, low velocity limit of the general theory of relativity. In addition, presenting this non-flat Newtonian spacetime is pedagogically valuable, as it gives learners of the general theory an introduction to necessary mathematical and geometric ideas independent of the initial conceptual difficulty of the full general theory of relativity. We also show that once we adopt a geometrized view of gravitation informed by the equivalence principle, we can predict the existence of magnetic-type gravitational interactions owing to the dragging of inertial frames in frames of reference in which any of the matter/energy content has a non-zero three-velocity, expounding upon Stachel 2006. We describe both the magnetic-type and electric-type inertiogravitational fields in the generalized case of a sphere of mass M and radius R rotating with angular velocity ω with respect to a non-rotating frame. We use this near-field, low-velocity model to predict that the amount of rotation the gryoscope in Stanford University/NASA's Gravity Probe B should be 24.9 mas/year in the direction opposite Earth's rotation.

1 Introduction - the Inertial, Motional Tendencies of Matter

Premised on the experimentally demonstrated equivalence of inertial and gravitational mass, the gravitational equivalence principle is a piece of scientific understanding that would have been conceptually meaningful even before the advent of the special theory of relativity. For once it is understood that locally, the effects of gravitation and those of being in an accelerated reference frame are indistinguishable, we must assign these two seemingly disparate processes the same physical significance.

Specifically, in traditional Newtonian theory, for a given gravitational field $\vec{g}$ and test particle,

$$m_I \vec{a} = m_G \vec{g}$$

, where $\vec{a}$ is the 3-acceleration of a test particle w.r.t. the center of mass of the matter giving rise to the gravitational field $\vec{g}$, m_I is a determiner of the 3-acceleration of test particle in response to a given external force, i.e. the test particle's inertial mass, and m_G is the gravitational mass or, in analogy with electromagnetism, gravitational "charge" of the test particle that determines the extent of the gravitational force acting on the test particle. Experimentally it has been shown to a high degree of precision that $\vec{a} = \vec{g}$ and thus $m_I = m_G$ for every test particle.

This forces us to reconceptualize the motional tendencies of matter as influenced by the force of gravity, in Newtonian theory, as merely the inertial motional tendencies of matter with respect to so-called inertial frames of reference in Newtonian theory. To be in a frame in which, locally, there is a so-called gravitational force must be reinterpreted as being in a frame which has a particular non-zero four-acceleration. In particular, we should require of such a theory that for a given point in the spacetime, the four-acceleration of a frame that has a zero 3-acceleration with respect to the center of mass of the source matter corresponds to what is described in traditional Newtonian theory as the gravitationally induced 3-acceleration of test particles w.r.t. the center of mass of the source matter.

First, to understand how we can even begin to formulate this statement mathematically, we must make a slight abstract change to how we usually think about time, space, and motion. In Newtonian mechanics, it is taken to be intuitive that time progresses at the same rate for all observers, and things move through space, and this motion through space is independent of the passage of time. However, we can, with equal validity, say that things occupy various positions in space *and* they occupy various 'positions' in time, i.e. different times. Although our concept of 'motion' is logically analzyable into a succession of places as a function of time, we can modify this concept and say that, in a different way, objects 'move' through the spacetime, occupying, throughout their histories, different places and different times. The collection of different places and different times an object is at throughout its history is called that object's *world line*. The abstract part of this definition is that it is a *timeless* statement that characterizes the object's motion, i.e. without reference to past, present, or future – all statements of the form "the object was at such and such a place then" is contained in the collection of the spaces and times that object occupies throughout its existence. Instead of saying the object "was at such and such a place then and will be at such and such a place later," we simply say that the object exists in a certain subset of the spacetime, this subset being the object's world line. We can then make the statement that the object moves through space and moves through time by introducing a parameter that corresponds to different points in the worldline as it increases, i.e. some function $X^\mu(\lambda)$ that gives the spatiotemporal situation of the object as a function of the parameter λ, where $X^0 = T = ct$ is the time of the object multiplied by some spacetime conversion factor to make all four coordinates of the object commensurate, and $X^i, i = 1, 2, 3$ are the three spatial coordinate positions of the object. The statement that the object moves through space and through time can then be made mathematically that the object has a *four-velocity* $W^\mu := \frac{dX^\mu}{d\lambda}$.

Such considerations do not require us to impose any 4-metric structure on the spacetime – this is only demanded by the need to construct an invariant spacetime interval under Lorentz transformations as a means of preserving the fundamental velocity of special relativity. This is itself a fundamentally different consideration. Thus, we are free to conceptually reformulate Newtonian theory with the above explicated insights. The way we will do this mathematically is the imposition of non-flat affine strucuture on a spacetime manifold. A spacetime manifold is a collection of spacetime points, i.e. times and places. In order to clarify how we can describe a manifold, we must first clarify what is meant by an affine space. A global affine space is an n-dimensional space that maintains all postulates of Euclidian geometry without any metric and hence no concept of length or angles between vectors. Formally, if we consider an n-dimensional vector space $\vec{U}$, and a n-dimensional set of points $\mathbf{A}$, $\mathbf{A}$ is an affine space if for every element $\vec{v}$ and $\vec{w}$ of $\vec{U}$ and for every element p of $\mathbf{A}$, then:

$$p + \vec{v} \in \mathbf{A}$$

$$(p + \vec{v}) + \vec{w} = p + (\vec{v} + \vec{w})$$

$$p + \vec{v} = p \leftrightarrow \vec{v} = \vec{0}$$

and for all p,q $\in \mathbf{A} \exists \vec{v}$ such that $\vec{v} = p - q$

An affine transformation $\mathbf{A} \rightarrow \mathbf{A'}$ preserves these properties, i.e. the coordinates of the new points in $\mathbf{A'}$, $X^{\gamma'}$, are related to the coordinates of the points in the original $\mathbf{A}$, X^γ, via

$$X^{\gamma'} = \frac{\partial X^{\gamma'}}{\partial X^\gamma} X^\gamma + C^\gamma$$

where $\frac{\partial X^{\gamma'}}{\partial X^\gamma}$ defines the transformation matrix, the $X^{\gamma'}$ are linear in the X^γ and the C^γ is a translation of the points. For our purposes, we will restrict ourselves to unimodular transformations, i.e. those which preserve n-volumes (4-volumes in a spacetime), in which $\left|\frac{\partial X^{\gamma'}}{\partial X^\gamma}\right| = 1$

In a global affine space, parallelism is a well-defined concept for any two vectors in the space. However, the most general description of a space, or, in our case, a spacetime, comes from the idea of a manifold, which is more general than a global affine space. What is particularly important for our case is that distant parallelism is not neccessarily a well-defined concept within the manifold. We can, however, describe vectors at a given point of the manifold in terms of the tangent space of that point of that manifold. A tangent space is the space spanned by the collectivity of the tangent vectors to any possible curve through that point. We can then take this locally defined tangent space to be centered affine, i.e. an affine space that has a fixed origin, in order to ensure that the space remains a description of the tangent space at the particular point of the manifold at which it is defined. This eliminates translations from the allowable coordinate transformations, i.e. we only consider $X^{\gamma'} = \frac{\partial X^{\gamma'}}{\partial X^{\gamma}} X^{\gamma}$ with $\left|\frac{\partial X^{\gamma'}}{\partial X^{\gamma}}\right| = 1$. In order to describe the relations of spacetime points in our manifold, we need to be able to connect neighboring tangent spaces. This we will do shortly.

We first turn to further characterize what we should require of our manifold. In order for this spacetime manifold to be Newtonian, we demand as conditions on the manifold that rate of any observer's passage through the 0th dimension, i.e. time, is the same, and that the 3-geometry of hypersurfaces that are transvected by the 0th dimension is affine flat, which is the same as to say Euclidean (although non-metrically).

We must now better define what we mean by an observer, which is closely tied to the idea of a frame of reference: an observer is at a particular point in a frame of reference. We can define a frame of reference globally, by either the spacetime coordinatization of all points in the manifold or with a field of tetrad vectors, i.e. a set of 4 vectors at every point of the manifold, which are linearly independent. We can locally define a frame of reference at a point by a coordinatization of the tangent space of the manifold or specification of a tetrad at that point, or of the tangent spaces at all points on a world-line, i.e. a curve through the spacetime. An observer can be described by such a curve through the spacetime. We are also free to define a local frame of reference by specifying either a tetrad at every point or coordinatization of the tangent space at every point of 2-dimensional hypersurfaces or 3-dimensional hypersurfaces of the manifold. In this sense, a frame of reference is local if it is not specified over the entirety of the spacetime manifold. Following our definition of reference frames in terms of tangent spaces or tetrad fields, we can also define a global frame of reference by a coordinatization of the tangent spaces of every point of the manifold. See Figures 1 and 2. Such a spacetime structure would contain the inertial frames of reference in the traditionally Newtonian sense. That is, frames, defined either locally or globally, in which there is no absolute rotation and zero three-acceleration w.r.t. the center of mass of the source matter.

Formally, we can introduce a four-dimensional manifold M that is topologically homeomorphic to R^4, and a foliation of M given by a differentiable function $T(X)$, where $T(X)$ represents the absolute time in Newtonian theory. We should require that T is commensurate with the 3 coordinates X, which are themselves taken to have spatial dimensions, and so relate it to the ordinary absolute Newtonian time t via $T = ct$, where for the moment the constant c has units of $\frac{[length]}{[time]}$ and merely serves as a conversion factor to make T spatially commensurate. There is no need at this point to interpret it physically. The level surfaces $T = const.$ define the spatial hyper-surfaces of (global) simultaneity. If we introduce the covector $T_\mu = \partial_\mu T$, we can distinguish a 4-vector B^μ as space-like if $B^\mu T_\mu = 0$, future time-like if $B^\mu T_\mu > 0$, and past time-like if $B^\mu T_\mu < 0$

If we are to allow the tangent spaces of the manifold to differ, then the ordinary concept of a directional derivative $\partial_\nu = \frac{\partial}{\partial X^\nu}$ is no longer meaningful, as for some vector B^μ at a point in the manifold, if we consider a coordinate transformation $X^{\gamma'} = \frac{\partial X^{\gamma'}}{\partial X^{\gamma}} X^{\gamma}$ then $\partial_\nu B^\mu$ is related to $\partial_{\nu'} B^{\mu'}$ via

$$\partial_{\nu'} B^{\mu'} = \partial_\nu B^\mu \frac{\partial X^{\mu'}}{\partial X^\mu} \frac{\partial X^\nu}{\partial X^{\nu'}} + (\frac{\partial^2 X^{\mu'}}{\partial X^\mu \partial X^{\nu'}}) B^\mu$$

, and $\frac{\partial^2 X^{\mu'}}{\partial X^\mu \partial X^{\nu'}}$ is in general non-zero if the tangent spaces differ, and thus $\partial_\nu B^\mu$ is not a tensor – in simpler terms, it is not enough to evaluate the change in the components of a vector in a given direction, as the very notion of what it means to be in a given direction changes from point to point in the manifold. We need to find an operation analogous to ∂_ν that accounts for differences in neighboring tangent spaces. Mathematically, we must search for an expression that is tensorial and a natural extension of ∂_ν. For readers unfamiliar with what it means for a quantity to be tensorial, it means that the $\frac{\partial^2 X^{\mu'}}{\partial X^\mu \partial X^{\nu'}}$ is not present in the expression for how the geometric object considered ($\partial_\nu B^\mu$ above – we are soon to find an adequate replacement) is described under a new system of coordinates. This is desirable because this term is a relic of the fact that the tangent spaces differ from point to point in the manifold.

It is easy to show that we can introduce an operation $D_\nu B^\mu$, called the covariant derivative, which is tensorial and related to $\partial_\nu B^\mu$ by a geometric object $\Gamma^\mu_{\kappa\nu}$ via $D_\nu B^\mu = \partial_\nu B^\mu + \Gamma^\mu_{\kappa\nu} B^\kappa$, where $\Gamma^\mu_{\kappa\nu}$ is required to have the transformation property of

$$\Gamma^{\mu'}_{\kappa'\nu'} = \Gamma^\mu_{\kappa\nu} \frac{\partial X^\nu}{\partial X^{\nu'}} \frac{\partial X^\kappa}{\partial X^{\kappa'}} \frac{\partial X^{\mu'}}{\partial X^\mu} - (\frac{\partial^2 X^{\mu'}}{\partial X^\kappa \partial X^\nu})(\frac{\partial X^{\kappa'}}{\partial X^\kappa})(\frac{\partial X^{\nu'}}{\partial X^\nu})$$

Conceptually, $\Gamma^\mu_{\kappa\nu}$ can be thought of as a geometric object connecting neighoring tangent spaces of the manifold, and so is called an affine connection. It may be helpful for understanding this object to think of it in terms of autoparallel transport – a vector in a manifold that 'moves itself' to the neighboring tangent (affine) space of the manifold. In the case of the surface of a sphere, perhaps the easiest to visualize non-flat manifold, a given vector autoparallel transporting itself defines the curve of a great circle on the sphere. If we consider a coordinatization in which a vector on the surface of the sphere takes itself around the so-called equator of the sphere ($\varphi = 0$ to $\varphi = 2\pi$), then the components of this vector will be constant – 0 in the θ (longitudinal) direction and some non-zero constant v^φ in the φ (latitudinal) direction. However, if we consider the same great circle after a change of coordinate system that rotates the so-called north pole of the sphere, then neither the θ nor the φ components of the autoparallel transported vector defining this curve will be constant, but the curve will still be just as much an autoparallel in the space. Thus, constancy of the coordinate components of a tangent vector is not a sufficent condition for defining an autoparallel in non-flat manifold. The introduction of $\Gamma^\mu_{\kappa\nu}$ makes the above two descriptions of the same autoparallel curve consistent, as it offers a description of parallel vectors from point to point in the manifold, i.e. it connects tangent spaces of the space, in a way that is independent of the coordinatization of that space. See Figure 3 for a visual demonstration of this argument.

In a spacetime manifold, then, the equations of motion that are invariant for all spacetime coordinatizations, i.e. all frames of reference, of a four-force-free test-particle with a four-velocity $W^\kappa := \frac{dX^\kappa}{d\lambda}$, where $X^\kappa(\lambda)$ are now the coordinates of the spacetime curve the particle follows, rather than the coordinates more generally, are given by the autoparallel condition:

$$W^\nu D_\nu W^\kappa = 0 \rightarrow$$

$$\frac{d^2X^\kappa}{d\lambda^2} + \Gamma^\kappa_{\nu\mu}\frac{dX^\mu}{d\lambda}\frac{dX^\nu}{d\lambda} = 0 \qquad (1)$$

,where λ is the preferred parameter of the curve. Conceptually, this states that W^ν parallel transports itself through the spacetime, and moves along this curve as we increase λ.

This expression has been traditionally called the geodesic equation. However, its conceptual introduction does not require any metric structure on the manifold, but rather only the idea of autoparallel transport in an affine space. Thus, it is most properly called the autoparallel equation. More generally, $W^\nu D_\nu W^\kappa = A^\kappa$ is the four-acceleration of a curve.

1.1 Newton's Three Laws

Newton's First law as he originally concieved it can now be thought of as the autoparallel condition in a globally flat affine spacetime, in which all $\Gamma^\mu_{\kappa\nu} = 0$ and $\frac{d^2X^\kappa}{d\lambda^2} = 0$ for some spacetime coordinate system X^μ (here referring to the coordinates themselves and not the coordinates of a curve), and additionally for every choice of coordinates related to this coordinate system in such a way that $(\frac{\partial^2 X^{\mu'}}{\partial X^\kappa \partial X^\nu})(\frac{\partial X^{\kappa'}}{\partial X^\kappa})(\frac{\partial X^{\nu'}}{\partial X^\nu}) = 0$. Physically, we can understand this as saying that the first coordinate system corresponds to some frame of reference in which Newton's First Law holds, i.e. some inertial frame of reference, and every coordinate system which is not accelerated w.r.t. this coordinate system, also corresponds to an inertial frame of reference in which all $\frac{d^2X^\kappa}{d\lambda^2} = 0$. See Figure 2. In this sense, Newton's First Law, at bottom, defines the frames of reference in which Newton's Second Law is valid. In geometrical language, these are the set of spacetime coordinate systems naturally adapted to the affine-flat structure of the spacetime. Newton's Second Law is then that the deviation of a particle's worldline from autoparallel transport in an affine-flat spacetime is proportional to the net force acting upon that particle and inversely proportional to the particle's mass: $a = \frac{F}{m}$. See Figure 3. Since the deviation from affine-flat autoparallel transport is the same for all particles being acted upon by the gravitational field, it is simpler to reconceptualize the deviation from affine-flat autoparallel transport as simply the new autoparallel transport of the spacetime. Newton's Third Law, combined with his first two, is then that the law of affine-flat autoparallel transport, i.e. momentum conservation, holds for the center of mass of a closed system. When the autoparallels are modified by gravitating matter, the center of mass of a closed system still follows autoparallel spacetime curves, although in curved spacetime, but a component of Newton's Third Law is reconceptualized: the recipricol nature of gravitational interactions then becomes simply that everything modifies the spacetime structure in exactly the same fashion (to be shown mathematically shortly), rather than two gravitating particles mutually pulling each other from autoparallel transport in an affine flat spacetime.

1.2 Non-Flat Manifolds

In the case where the manifold is non-flat, we are always free to pick a frame of reference along a test particle's world-line which is co-accelerating with the test particle, in which case, locally $\frac{d^2X^\kappa}{d\lambda^2}=0$ and thereby all $\Gamma^\kappa_{\nu\mu}=0$. Indeed, as we shall see, this defines the only frame of reference which resembles the concept of a Newtonian inertial frame – it differs in that there still must be an observed relative acceleration of test-particles and the center of mass of the source matter, and that in the infinitesimal neighborhood of any point for which we have locally considered this co-accelerating frame, there must exist so-called tidal forces, a result of the non-zero curvature of the spacetime and described in geometrical terms by the equation of autoparallel deviation, which will be elaborated upon in coming sections. More generally, the autoparallel condition, as given by equation (1), is always true, but it is not always true that $\Gamma^\kappa_{\nu\mu}=0$ *and* $\frac{d^2X^\kappa}{d\lambda^2}=0$. That is, one can, by a suitable transformation of coordinates, change what was attributed to the gravitational force in one frame as being attributed as merely inertia in transformed frame. While in Newtonian theory, it is posited that there is an underlying inertial structure, and a gravitational field deflecting particles from what would be otherwise inertial tendencies, the gravitational equivalence principle informs us that these are one and the same structure on the spacetime. Instead of being characterized independently by inertial structure and a gravitational field, there is inertiogravitational structure, or an inertiogravitational field if you rather, on the spacetime, definable rigorously in terms of the affine connection.

Now we introduce a triad of spatial basis vector fields $e^\mu_{(i)}$, with a dual co-basis $e^{(i)}_\mu$ defined by $e^\nu_{(j)}e^{(i)}_\nu=\delta^{(i)}_{(j)}$ and $e^\nu_{(i)}e^{(i)}_\mu=\delta^\nu_\mu$ and that span the tangent space and cotagent space at each point of each leaf of the foliation. We then define a time-like vector field that transvects the spatial hypersurfaces and thereby defines a global frame of reference, $e^\mu_{(0)}$, with corresponding co-basis vector field $e^{(0)}_\mu := \partial_\mu T$ such that $e^\nu_{(0)}e^{(0)}_\nu=1$. This completes duality conditions between the basis and co-basis, i.e. $e^\nu_{(\beta)}e^{(\alpha)}_\nu=\delta^{(\alpha)}_{(\beta)}$ and $e^\nu_{(\alpha)}e^{(\alpha)}_\mu=\delta^\nu_\mu$ Note that the (i) or (j) merely indicates enumeration from 1 to 3, while the (α) or (β) indicates enumeration from 0 to 3.

We can then project W^κ onto this tetrad, $W^\kappa=W^{(\alpha)}e^\kappa_{(\alpha)}$, with $W^{(0)}=1$ and $W^{(i)}=\frac{w^{(i)}}{c}$, i.e. the passage of time is the same for the test-particle as it is for the $e^\nu_{(0)}$ which defines the frame of reference, and the test-particle has 3-velocity components of $\frac{w^{(i)}}{c}$ along spatial contravector $e^\mu_{(i)}$. Making the scalars $W^{(\alpha)}$ unitless ensures that all components of W^κ have spatial units, as the tetrad has spatial units. The autoparallel condition then becomes

$$(W^{(\alpha)}e^\nu_{(\alpha)})D_\nu(W^{(\beta)}e^\kappa_{(\beta)})=0\rightarrow$$

$$W^{(\alpha)}e^\nu_{(\alpha)}(D_\nu W^{(\beta)})e^\kappa_{(\beta)}+e^\nu_{(\alpha)}D_\nu e^\kappa_{(\beta)}W^{(\alpha)}W^{(\beta)}=0$$

If we contract this with $e^{(\gamma)}_\kappa$ we have

$$W^{(\alpha)}e^\nu_{(\alpha)}(D_\nu W^{(\beta)})e^\kappa_{(\beta)}e^{(\gamma)}_\kappa+e^{(\gamma)}_\kappa e^\nu_{(\alpha)}D_\nu e^\kappa_{(\beta)}W^{(\alpha)}W^{(\beta)}=0$$

or, defining $\frac{D}{d\lambda} := W^\nu D_\nu$ and the tetrad components of the affine connection, or t.c.c. for short, as $\Gamma^{(\gamma)}_{(\alpha)(\beta)} := e^{(\gamma)}_\kappa e^\nu_{(\alpha)} D_\nu e^\kappa_{(\beta)}$

$$\left(\frac{DW}{d\lambda}\right)^{(\gamma)} + \Gamma^{(\gamma)}_{(\alpha)(\beta)} W^{(\alpha)} W^{(\beta)} = 0 \qquad (2)$$

Conceptually, the first term is the change in four-velocity of the test particle w.r.t. the frame defined by the tetrad, due to W^ν parallel transporting itself, projected upon the co-basis tetrad $e^{(\gamma)}_\kappa$, hence enclosing the whole first term associated with index (γ) in parentheses. Numerically, however, owing to the imposed duality of the basis, it is equal to the change in scalar $W^{(\gamma)}$ due to W^ν parallel transporting itself. Physically the four scalars $\left(\frac{DW}{d\lambda}\right)^{(\gamma)}$ describe, in the frame of reference defined by the tetrad, the 3-acceleration of the four-force-free test particle as well as its change w.r.t. the affine parameter of the temporal coordinate of the test-particle's four-velocity, the latter of which should be zero by our compatibility conditions.

This approach has the advantage that the set of 64 scalars $\Gamma^{(\gamma)}_{(\alpha)(\beta)}$ are locally meaningful for a given basis: the result of any measurement is a scalar, and so describing the spacetime structure by a set of sixty-four scalars dependent on the choice of directions (both spatial and temporal) considered for an experiment makes them meaningful as scalars related to the result of a measurement.

We now turn to the compatibility conditions between the connection and the chronometry (i.e. absolute time) and the spatial basis vectors, which require Euclidean 3-geometry of the spatial hypersurfaces, to see which t.c.c. are eliminated a priori. To require that absolute time holds for all frames of reference is equivalent to demanding that $D_\nu T_\mu = 0 \leftrightarrow D_\nu e^{(0)}_\mu = 0$, i.e. the magnitude of the covector defining the temporal separation of the spatial hypersurfaces does not change at any point of the manifold in the direction of any of the vectors.

Now we show that

$$\Gamma^{(0)}_{(\alpha)(\beta)} = e^{(0)}_\kappa e^\nu_{(\alpha)} D_\nu e^\kappa_{(\beta)} = e^\nu_{(\alpha)} D_\nu (e^{(0)}_\kappa e^\kappa_{(\beta)}) - e^\kappa_{(\beta)} e^\nu_{(\alpha)} D_\nu e^{(0)}_\kappa = e^\nu_{(\alpha)} D_\nu (\delta^{(0)}_{(\beta)}) - e^\kappa_{(\beta)} e^\nu_{(\alpha)} D_\nu e^{(0)}_\kappa = 0$$

by the duality conditions of the basis and the compatibility condition requiring an absolute time.

We now impose the Euclidicity of the 3-spaces of the spatial hypersurfaces by demanding that parallel transport constrained to such a hypersurface be independent of the space-like path. Formally we can thus require

$$e^\nu_{(a)} D_\nu e^\kappa_{(b)} = 0$$

from which it follows that $\Gamma^{(\gamma)}_{(a)(b)} = 0$.

Thus requiring absolute Newtonian time and flat Euclidean 3-space only allows for non-zero $\Gamma^{(a)}_{(0)(0)}$, $\Gamma^{(a)}_{(0)(b)}$, and $\Gamma^{(a)}_{(b)(0)}$ and demands that all others be zero.

Because we require that the spatial triad remain dual, we require that

$$e^{(a)}_\kappa D_\nu e^\kappa_{(b)} = -e^{(b)}_\kappa D_\nu e^\kappa_{(a)}$$

and thus

$$e^{(a)}_\kappa e^\nu_{(0)} D_\nu e^\kappa_{(b)} = -e^{(b)}_\kappa e^\nu_{(0)} D_\nu e^\kappa_{(a)} \rightarrow$$

$$\Gamma^{(a)}_{(0)(b)} = -\Gamma^{(b)}_{(0)(a)}$$

With our compatibility conditions imposed on the t.c.c., if we now look at the (0) component of equation (2), we have

$$\left(\frac{DW}{d\lambda}\right)^{(0)} + \Gamma^{(0)}_{(\alpha)(\beta)} W^{(\alpha)} W^{(\beta)} = 0$$

thus implying

$$\left(\frac{DW}{d\lambda}\right)^{(0)} = 0$$

Thus the change in the passage of time for the test particle via parallel transport by its own W^{ν} defined in reference to the preferred parameter λ is always zero. In more mathematical terms, λ agrees with the absolute time T up to a linear rescaling and choice of origin. For simplicity, therefore, we choose to identify the two, thereby now defining $W^{\nu} := \frac{dX^{\nu}}{dT}$. From the fact that all $\Gamma^{(0)}_{(\alpha)(\beta)} = 0$, it follows that all $\Gamma^{0}_{\mu\nu} = 0$ and thus, from equation (1), $\frac{d^2 X^0}{d\lambda^2} = 0$. Thus, we see that in the same way that we are free to identify the preferred parameter λ with absolute Newtonian time T, we are free to identify the 0th coordinate in the spacetime manifold as $X^0 = \lambda = T$.

The equations of four-force-free motion now become

$$\left(\frac{DW}{dT}\right)^{(m)} + \Gamma^{(m)}_{(0)(0)} + \Gamma^{(m)}_{(0)(n)} W^{(n)} + \Gamma^{(m)}_{(n)(0)} W^{(n)} = 0$$

,

or, w.r.t. traditional absolute time t

$$\frac{1}{c^2}\left(\frac{Dw}{dt}\right)^{(m)} + \frac{1}{c^2}\Gamma^{(m)}_{(t)(t)} + \frac{1}{c}\Gamma^{(m)}_{(t)(n)} \frac{w^{(n)}}{c} + \frac{1}{c}\Gamma^{(m)}_{(n)(t)} \frac{w^{(n)}}{c} = 0$$

after cancelling out the common $\frac{1}{c^2}$ term we have

$$\left(\frac{Dw}{dt}\right)^{(m)} + \Gamma^{(m)}_{(t)(t)} + \Gamma^{(m)}_{(t)(n)} w^{(n)} + \Gamma^{(m)}_{(n)(t)} w^{(n)} = 0 \tag{3}$$

As we will see, the second term corresponds to the Newtonian electric-type gravitational force, while the third and fourth, 3-velocity dependent terms correspond to a magnetic-type gravitational interaction. The $\Gamma^{(m)}_{(0)(n)}$ and the $\Gamma^{(m)}_{(n)(0)}$ are related by the relation between the t.c.c., the torsion tensor, and the anholonomic object given by Papapetrou-Stachel 1978. In the case of a torsionless connection, this reduces to

$$\Gamma^{(\gamma)}_{[(\alpha)(\beta)]} = -\Omega^{(\gamma)}_{(\alpha)(\beta)} \tag{4}$$

where

$$\Omega^{(\gamma)}_{(\alpha)(\beta)} = \frac{1}{2} e^{\nu}_{(\alpha)} e^{\mu}_{(\beta)} (e^{(\gamma)}_{\mu,\nu} - e^{(\gamma)}_{\nu,\mu})$$

is the anholonomic object of the basis.

Note that, if we consider a holonomic basis, with all $\Omega^{(\gamma)}_{(\alpha)(\beta)} = 0$, then equation (4) tells us

$$\Gamma^{(m)}_{(0)(n)} = \Gamma^{(m)}_{(n)(0)}$$

Since our compatibility conditons have imposed that

$$\Gamma^{(m)}_{(0)(n)} = -\Gamma^{(n)}_{(0)(m)}$$

,

it follows in an holonomic spacetime that

$$\Gamma^{(m)}_{(n)(0)} = -\Gamma^{(n)}_{(m)(0)}$$

.

Visually, it is easy to see that such a change is a rotation of the spatial triads $e^{\kappa}_{(m)}$ and $e^{\kappa}_{(n)}$, along with the corresponding dual covectors $e^{(m)}_{\kappa}$ and $e^{(n)}_{\kappa}$. See Figure 6. This change in spatial triad occurs in the direction of the manifold defined by $e^{\nu}_{(0}$, which goes from hypersurface to hypersurface. Thus the $\Gamma^{(m)}_{(n)(0)}$ and $\Gamma^{(m)}_{(0)(n)}$ represent a rotation *rate* in a holonomic spacetime.

We can now describe the Coriolis and centrifugal forces quite naturally from this geometric description of the spacetime. We begin with a flat affine spacetime with all $\Gamma^{(\gamma)}_{(\alpha)(\beta)} = 0$, and then transform to a frame of reference rotating with constant $\vec{\omega} = \omega e^{\nu}_{(3)}$. Our rotating frame of reference is related to the first via

$$e'^{\nu}_{(0)} = e^{\nu}_{(0)} + \frac{\vec{\omega}}{c} \times \vec{r} = e^{\nu}_{(0)} + \frac{\omega x^{(1)}}{c} e^{\nu}_{(2)} - \frac{\omega x^{(2)}}{c} e^{\nu}_{(1)}$$

, where $\vec{r} = x^{(m)} e^{\nu}_{(m)}$ and $e^{\nu}_{(m)} = \delta^{\nu}_{(m)}$,

$$e'^{\nu}_{(1)} = cos\frac{\omega T}{c} e^{\nu}_{(1)} + sin\frac{\omega T}{c} e^{\nu}_{(2)}$$

$$e'^{\nu}_{(2)} = -sin\frac{\omega T}{c} e^{\nu}_{(1)} + cos\frac{\omega T}{c} e^{\nu}_{(2)}$$

$$e'^{\nu}_{(3)} = e^{\nu}_{(3)}$$

$$e'^{(1)}_{\nu} = cos\frac{\omega T}{c} e^{(1)}_{\nu} + sin\frac{\omega T}{c} e^{(2)}_{\nu}$$

$$e'^{(2)}_{\nu} = -sin\frac{\omega T}{c} e^{(1)}_{\nu} + cos\frac{\omega T}{c} e^{(2)}_{\nu}$$

$$e'^{(3)}_{\nu} = e^{(3)}_{\nu}$$

.

Recalling that we have chosen to identify x^0 and T, it follows that

$$\Gamma'^{(1)}_{(0)(0)} = -\frac{\omega^2}{c^2}(x^{(1)} cos\frac{\omega T}{c} + x^{(2)} sin\frac{\omega T}{c})$$

$$\Gamma'^{(2)}_{(0)(0)} = -\frac{\omega^2}{c^2}(-x^{(1)} sin\frac{\omega T}{c} + x^{(2)} cos\frac{\omega T}{c}) \rightarrow$$

$$\Gamma^{'(1)}_{(t)(t)} = -\omega^2(x^{(1)}cos\omega t + x^{(2)}sin\omega t) = -\omega^2 x^{'(1)}$$

$$\Gamma^{'(2)}_{(t)(t)} = -\omega^2(-x^{(1)}sin\omega t + x^{(2)}cos\omega t) = -\omega^2 x^{'(2)}$$

i.e. the $\Gamma^{'(1)}_{(t)(t)}$ and $\Gamma^{'(2)}_{(t)(t)}$ are the (negative) of the centrifugal acceleration, $\vec{a}_c = \omega^2(x\hat{x} + y\hat{y})$, projected upon the rotating basis of vectors.

We also have

$$\Gamma^{'(1)}_{(0)(2)} = \Gamma^{'(1)}_{(2)(0)} = \frac{\omega}{c} \leftrightarrow \Gamma^{'(1)}_{(t)(2)} = \Gamma^{'(1)}_{(2)(t)} = \omega$$

$$\Gamma^{'(2)}_{(0)(1)} = \Gamma^{'(2)}_{(1)(0)} = -\frac{\omega}{c} \leftrightarrow \Gamma^{'(2)}_{(t)(1)} = \Gamma^{'(2)}_{(1)(t)} = -\omega$$

Looking at the equations of motion for a four-force free test particle in this frame of reference,

$$\frac{D^2 x'^{(1)}}{dt^2} = -\Gamma^{(1)}_{(t)(t)} - \Gamma'^{(1)}_{(t)(2)} w'^{(2)} - \Gamma'^{(1)}_{(2)(t)} w'^{(2)}$$

$$\frac{D^2 x'^{(2)}}{dt^2} = -\Gamma^{(2)}_{(t)(t)} - \Gamma'^{(2)}_{(t)(1)} w'^{(1)} - \Gamma'^{(2)}_{(1)(t)} w'^{(1)}$$

$$\frac{D^2 x'^{(3)}}{dt^2} = 0$$

we see that they agree with the usual description of the centrifugal forces and Coriolis forces as $\vec{F}_{CF} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r})$ and $\vec{F}_{COR} = -2m\vec{\omega} \times \vec{v}$. The interpretation of the factor of 2 in the Coriolis force immediately follows from this formalism: one factor comes from the fact that the rotating reference frame's basis vectors are rotating w.r.t. the inertial reference frame, i.e. from the $\Gamma'^{(x)}_{(t)(y)} = \omega$ and the $\Gamma'^{(y)}_{(t)(x)} = -\omega$, and the other comes from the fact that the spatial component of the $e'^{\nu}_{(0)}$, i.e. the tangent velocity of the rotating reference frame itself, varies from point to point in the manifold, i.e. from the $\Gamma'^{(x)}_{(y)(t)} = \omega$ and the $\Gamma'^{(y)}_{(x)(t)} = -\omega$. This same analysis could have been performed with a cylindrical-polar set of spatial basis vectors, however doing so would have introduced non-zero $\Gamma^{\nu}_{\mu\kappa}$ and hence non-zero $\Gamma^{(\gamma)}_{(\alpha)(\beta)}$ in the non-rotating reference frame, even though the spacetime on which the tetrad is defined is postulated to be affine flat. We present here the derivation in terms of the Cartesian spatial basis vectors $\delta^{\kappa}_{(m)}$ because it shows how the centrifugal and Coriolis forces arise strictly due to the relation of the rotating reference frame to the underlying affine flat chronogeometric structure of the spacetime.

2 Geometrized Gravitational Dynamics - General Case

Once we have adopted a spacetime view informed by the gravitational equivalence principle with the above affine structure, in order to reformulate Newton's gravitational theory, the question becomes: how do we treat matter's influence on the spacetime structure? We have already seen that the t.c.c. are not invariantly meaningful, as in a frame with zero 3-acceleration w.r.t. the source matter, we expect that they roughly correspond to the 3-acceleration due to the so-called gravitational force in traditional Newtonian gravitational theory (I say roughly because additional considerations to follow offer an extension of, rather than merely a reformulation of, Newton's equations of motion in the presence of gravitation), while in a frame that is locally co-accelerating with a test-particle, they vanish. Once it is understood that it is actually the affine curvature of the spacetime that is invariant for all observers, regardless of their relative 3-accelerations or absolute 4-accelerations, we must attempt to formulate field equations for the spacetime in terms of the non-vanishing components of the affine curvature tensor, the values of which must vary differentiably and be determined by the momentum-energy distribution in the spacetime. Traditional Newtonian gravitational theory, with non-vanishing $\Gamma^{(m)}_{(0)(0)}$, would lead us to

$$R_{(0)(0)} = (const.)G\rho$$

where $R_{(0)(0)}$ is the (0)(0) tetrad component of the affine Ricci tensor, given by Papapetrou-Stachel 1978 as

$$R_{(\lambda)(\mu)} = \Gamma^{(\kappa)}_{(\lambda)(\mu),(\kappa)} - \Gamma^{(\kappa)}_{(\kappa)(\mu),(\lambda)} + \Gamma^{(\kappa)}_{(\kappa)(\rho)}\Gamma^{(\rho)}_{(\lambda)(\mu)} - \Gamma^{(\kappa)}_{(\lambda)(\rho)}\Gamma^{(\rho)}_{(\kappa)(\mu)} + 2\Omega^{(\rho)}_{(\kappa)(\lambda)}\Gamma^{(\kappa)}_{(\rho)(\mu)}$$

where $_{,(\kappa)} = \partial_{(\kappa)} := e^{\rho}_{\kappa}\partial_{\rho}$.

Following from the fact that we have eliminated all but the $\Gamma^{(m)}_{(0)(0)}$, $\Gamma^{(m)}_{(0)(n)}$, and $\Gamma^{(m)}_{(0)(n)}$, we have

$$R_{(0)(0)} = \Gamma^{(m)}_{(0)(0),(m)}$$

$$R_{(0)(n)} = \Gamma^{(m)}_{(0)(n),(m)}$$

$$R_{(n)(0)} = \Gamma^{(m)}_{(n)(0),(m)} + 2\Omega^{(\rho)}_{(m)(n)}\Gamma^{(m)}_{(\rho)(0)}$$

$$R_{(m)(n)} = 0$$

If we assume the spacetime to be holonomic, we have $\Gamma^{(m)}_{(0)(n)} = \Gamma^{(m)}_{(n)(0)}$ and also

$$R_{(n)(0)} = \Gamma^{(m)}_{(n)(0),(m)} = R_{(0)(n)}$$

Thus our theory also permits non-zero $R_{(0)(n)}$ and $R_{(n)(0)}$, which are equal for a holonomic spacetime.

It is natural to assume that, in this case, these are given, analogous to magnetic-type Einstein field equations $G_{0n} = \frac{8\pi G}{c^4}T_{0n}$, by

$R_{(0)(n)} = (const.)\rho V^{(n)} = (const.)\rho\frac{v^{(n)}}{c}$

where we have the same constant as in our expression for $R_{(0)(0)}$ and $V^{(n)} = \frac{v^{(n)}}{c}$ is the 3-velocity of the source-matter.

Following Stachel 2006, if we assume that the $\Gamma^{(m)}_{(0)(0)}$ and the $\Gamma^{(m)}_{(0)(n)}$ are derivable from a gravitational scalar potential φ and gravitational vector potential $\vec{A}$, respectively, namely

$$\Gamma^{(m)}_{(0)(0)} = \delta^{(m)(j)}\partial_{(j)}\varphi \tag{5}$$

and

$$\Gamma^{(m)}_{(0)(n)} = \delta^{(m)(j)}[\partial_{(j)}A_{(n)} - \partial_{(n)}A_{(j)}], \tag{6}$$

and take $const. = \frac{4\pi}{c^2}$ in the field equations, they reduce to

$$R_{(0)(0)} = \delta^{mj}\partial_{mj}\phi = \nabla^2\varphi = \frac{4\pi G\rho}{c^2} \tag{7}$$

and

$$R_{(0)(n)} = \delta^{mj}\partial_{mj}A_{(n)} = \nabla^2 A_{(n)} = \frac{4\pi G\rho v^{(n)}}{c^3} \tag{8}$$

with the condition $\partial_j(\delta^{mj}A_m) = \nabla\cdot\vec{A} = 0$, for the case of a holonomic spacetime.

3 Electric-type Inertiogravitational Field of Earth

We now consider the magnetic-type and electric-type t.c.c. for a simplified model of the Earth: a rigid, uniform sphere of radius R, mass M, and constant angular velocity $\vec{\omega}$.

We begin with the $\Gamma^{(m)}_{(0)(0)}$. If we work in spherical polar coordinates (r, θ, ϕ) the invariant equation $\nabla^2\varphi = \frac{4\pi G\rho}{c^2}$ gives, most generally, the following solution for φ:

$$\varphi = \frac{1}{c^2}\begin{cases} \frac{GMr^2}{2R^3} + \frac{C_r}{r} + C_\phi\phi + C_\theta ln(cot\theta + csc\theta) + C & \text{if } r \leq R \\ -\frac{GM}{r} + \frac{C_r}{r} + C_\phi\phi + C_\theta ln(cot\theta + csc\theta) + C & \text{if } r \geq R \end{cases}$$

.

A quick calculation shows that this does indeed give $\nabla^2\varphi = \frac{4\pi G\rho}{c^2}$ inside the sphere and $\nabla^2\varphi = 0$ outside the sphere, while $\nabla\varphi$ is continuous, satisfying the demand that φ be differentiable. In the interest of ease of conceptual interpretation for the forthcoming calculation, we momentarily work with the solution for φ in Cartesian coordinates:

$$\varphi = \frac{1}{c^2}\begin{cases} \frac{GMx^{(i)}x^{(i)}}{2R^3} + a_{(i)}x^{(i)} + C & \text{if } x^{(i)}x^{(i)} \leq R^2 \\ -\frac{GM}{(x^{(i)}x^{(i)})^{\frac{1}{2}}} + a_{(i)}x^{(i)} + C & \text{if } x^{(i)}x^{(i)} \geq R^2 \end{cases}$$

,

where the $a_{(i)}$ are a set of 3 constants. We now evaluate the $\Gamma^{(m)}_{(t)(t)}$:

$$\Gamma^{(m)}_{(t)(t)} = \begin{cases} \frac{GMx^{(m)}}{R^3} + a_{(m)} & \text{if } x^{(i)}x^{(i)} \leq R^2 \\ \frac{GMx^{(m)}}{(x^{(i)}x^{(i)})^{\frac{3}{2}}} + a_{(m)} & \text{if } x^{(i)}x^{(i)} \geq R^2 \end{cases} \tag{9}$$

It is clear by inspection that the $a_{(m)}$ are the 3-accelerations of a frame of reference w.r.t. the center of mass of the source mass. As 3-acceleration entails a changing 3-velocity as a function of time, this 3-acceleration can be defined locally along a world-line or globally, as these are the two definitions of a frame of reference which are defined for more than one time. From the definition of the $\Gamma^{(m)}_{(0)(0)} = e^{(m)}_\kappa e^\nu_{(0)} D_\nu e^\kappa_{(0)}$ we see that they are the four-acceleration of such a locally defined frame of reference, $A^\kappa = e^\nu_{(0)} D_\nu e^\kappa_{(0)}$, projected upon the spatial triad of covectors. Thus we may say that objects resting on the surface of a rigid non-rotating sphere have a four-acceleration of $A^\kappa = \frac{GM}{c^2R^2}e^\kappa_{(r)}$. In order to see what the four-force per unit mass required to achieve this four-acceleration w.r.t. ordinary time t is, we must multiply by c^2, telling us that the required four-force per unit mass to keep an object resting on the surface of a rigid non-rotating sphere of mass M and radius R is $\frac{F^\kappa}{m} = \frac{GM}{R^2}e^\kappa_{(r)}$. In the spherical polar case, as well, the extra terms in φ correspond to a freedom to select a 3-accelerating frame of reference, and it is merely the form of the spherical polar Laplacian that requires the convoluted functional forms they take. If we define a global frame of reference with all $a_{(m)} = 0$ throughout the manifold, i.e. a global frame of reference with zero three-acceleration w.r.t. the center of mass of the source mass (Earth), Figure 6 shows the field of the local four-force per unit mass required to be in such a frame.

If we select a global frame of reference with non-zero $a_{(m)}$ with no positional dependence, we can see that the global pattern of the relative accelerations of the four-force-free test-bodies is an invariant for any such global frame of reference. I.e., for some point in the manifold $\vec{X}_0$

$$\Gamma^{(m)}_{(t)(t)}(\vec{X}) - \Gamma^{(m)}_{(t)(t)}(\vec{X})|_{\vec{X}=\vec{X}_0}$$

is an invariant expression for any global frame of reference with non-zero $a_{(m)}$ with no positional dependence, and can be visualized in this particular case for $\vec{X}_0 = \vec{0}$ with Figure 7. Mathematically, this is the statement that, by the definition of the affine connection, the difference of two connections is a tensor, and hence the difference of two tetrad components of the connection is a frame invariant.

Figure 6 also serves as a visualization of the tidal forces – for if the equations of motion, neglecting the much smaller magnetic-type effects to be examined, are given by $\frac{d^2x^{(m)}}{dt^2} = -\Gamma^{(m)}_{(t)(t)}$, the local variation in the magnitudes of the $\Gamma^{(m)}_{(t)(t)}$, as shown by this figure will give the relative acceleration of test-particles in the local neighborhood of a point – the tidal forces in traditional Newtonian theory. Thus, even if we choose a freely falling frame, there will be a deviation in spacetime paths of test particles in the local neighborhood of the freely falling frame. This relative acceleration can be found by considering an autoparallel curve $X^\mu(\lambda)$ and a neighboring autoparallel curve connected by a separation vector $\sigma^\mu(\lambda)$, where $\sigma^\mu(\lambda)$ is small compared to the distance scale on which the affine connection's variation throughout the manifold depends, so that $X^\mu(\lambda)+\sigma^\mu(\lambda)$ is also an autoparallel curve, and $W^\mu = \frac{dX^\mu}{d\lambda}$ as well as $N^\mu = W^\mu + \frac{d\sigma^\mu}{d\lambda}$ satisy the autoparallel transport condition, i.e.

$$\frac{dW^\mu}{d\lambda} + \Gamma^\mu_{\gamma\nu} W^\gamma W^\nu = 0 \tag{10}$$

, where the $\Gamma^\mu_{\gamma\nu}$ is evaluated along the $X^\mu(\lambda)$ curve, and

$$\frac{dW^\mu}{d\lambda} + \frac{d^2\sigma^\mu}{d\lambda^2} + \Gamma^\mu_{\gamma\nu}|_{X^\mu(\lambda)+\sigma^\mu(\lambda)}(W^\gamma + \frac{d\sigma^\gamma}{d\lambda})(W^\nu + \frac{d\sigma^\nu}{d\lambda}) = 0 \tag{11}$$

An expression for the observed relative acceleration of the two curves can be found by twice taking the W^ν covariant directed derivative of σ^μ, i.e. $T^\mu = \frac{D^2\sigma^\mu}{d\lambda^2}$ where $\frac{D}{d\lambda} = W^\nu D_\nu$:

$$V^\mu = W^\nu D_\nu \sigma^\mu = \frac{d\sigma^\mu}{d\lambda} + \Gamma^\mu_{\gamma\nu}\sigma^\gamma W^\nu \tag{12}$$

$$T^\mu = W^\epsilon D_\epsilon V^\mu = \frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\delta\epsilon} V^\delta W^\epsilon \tag{13}$$

Substituting (12) into (13) we have

$$T^\mu = \frac{d}{d\lambda}(\frac{d\sigma^\mu}{d\lambda} + \Gamma^\mu_{\gamma\nu}\sigma^\gamma W^\nu) + \Gamma^\mu_{\delta\epsilon}(\frac{d\sigma^\delta}{d\lambda} + \Gamma^\delta_{\gamma\nu}\sigma^\gamma W^\nu)W^\epsilon \rightarrow$$

$$T^\mu = \frac{d^2\sigma^\mu}{d\lambda^2} + \frac{d}{d\lambda}\Gamma^\mu_{\gamma\nu}\sigma^\gamma W^\nu + 2\Gamma^\mu_{\gamma\nu}\frac{d\sigma^\gamma}{d\lambda}W^\nu + ...$$

$$\Gamma^\mu_{\gamma\nu}\sigma^\gamma\frac{dW^\nu}{d\lambda} + \Gamma^\mu_{\delta\epsilon}\Gamma^\delta_{\gamma\nu}\sigma^\gamma W^\nu W^\epsilon$$

where we have used $\Gamma^\mu_{\gamma\nu} = \Gamma^\mu_{\nu\gamma}$.

$\frac{dW^\nu}{d\lambda}$ can be found with equation (10), we may set $\frac{d}{d\lambda}\Gamma^\mu_{\gamma\nu} = \Gamma^\mu_{\gamma\nu,\epsilon}W^\epsilon$, and $\frac{d^2\sigma^\mu}{d\lambda^2}$ can be found by expanding equation (11):

First we note that, since σ^μ is small, we may set $\Gamma^\mu_{\gamma\nu}|_{X^\mu(\lambda)+\sigma^\mu(\lambda)} = \Gamma^\mu_{\gamma\nu,\epsilon}\sigma^\epsilon$. Additionally, if σ^μ is to remain small, $\frac{d\sigma^\mu}{d\lambda}$ must also be small. Thus we may drop terms of the form $\sigma^\epsilon\frac{d\sigma^\mu}{d\lambda}$, as they are doubly small. Then, after expanding equation (11) and using equation (10) to eliminate appropriate terms in this expansion, we have

$$\frac{d^2\sigma^\mu}{d\lambda^2} = -2\Gamma^\mu_{\gamma\nu}\frac{d\sigma^\gamma}{d\lambda}W^\nu - \Gamma^\mu_{\gamma\nu,\epsilon}\sigma^\epsilon W^\gamma W^\nu$$

.

Substituting all these into our expression for T^μ, rearranging terms, and renaming dummy indices we have

$$T^\mu = (-\Gamma^\mu_{\epsilon\nu,\gamma} + \Gamma^\mu_{\gamma\nu,\epsilon} - \Gamma^\mu_{\gamma\kappa}\Gamma^\kappa_{\nu\epsilon} + \Gamma^\nu_{\delta\epsilon}\Gamma^\delta_{\gamma\nu})W^\nu W^\epsilon\sigma^\gamma \quad (14)$$

The expression modifying $W^\nu W^\epsilon\sigma^\gamma$, happens to be the negative of the affine curvature tensor, so we may write

$$\frac{D^2\sigma^\mu}{d\lambda^2} = -A^\mu_{\nu\epsilon\gamma}W^\nu W^\epsilon\sigma^\gamma$$

This is most properly called the equation of autoparallel deviation, as it is derivable in this curved spacetime by determining the evolution of a separation vector of two autoparallel curves in curved affine space.

The affine curvature tensor, $A^\kappa_{\nu\epsilon\gamma}$, is defined by

$$B_\mu A^\mu_{\nu\epsilon\gamma} = D_\epsilon D_\gamma B_\nu - D_\gamma D_\epsilon B_\nu$$

,

which is a local measure of the path-dependence of the parallel transport of a vector, a result of the non-zero curvature of the manifold – see Figure 8. It is again easy to visualize that there should be a path dependence of the parallel transport of a vector in a curved manifold if we consider parallel transporting a vector around a loop on the surface of a sphere. See Figure 9. The surface of a sphere also affords us a convenient visual aid for why there should be an evolution of the separation vector of two neighboring autoparallels in a curved manifold. See Figure 10. For these visual aids, however, it should be noted that the surface of a sphere is a manifold of constant curvature, while a spacetime in general has curvature which varies from point to point in the manifold.

If we, for the moment, neglect the magnetic-type gravitational interactions to be considered shortly, which, as we will see, are much smaller in magnitude, we can easily adapt spacetime coordinates to the above basis, in which case the tetrad components of the connection as given by equation (9) are equal to the coordinate components of the connection, i.e.

$$\Gamma^m_{00} = \begin{cases} \frac{GMx^{(m)}}{c^2R^3} + a_{(m)} & \text{if } x^{(i)}x^{(i)} \leq R^2 \\ \frac{GMx^{(m)}}{c^2(x^{(i)}x^{(i)})^{\frac{3}{2}}} + a_{(m)} & \text{if } x^{(i)}x^{(i)} \geq R^2 \end{cases}$$

In evaulating the tidal forces, we choose to continue to work in Cartesian spatial coordinates, as we are attempting to relate local coordinates, i.e. those around the so-called fiducial autoparallel (with $\sigma^\mu = 0$), to global coordinates describing the variation in the manifold of the connection. If we now consider autoparallel curves limited to the equatorial cross-section, using equation (14) we have

$$T^1 = \begin{cases} -\Gamma^1_{00,1}W^0W^0\sigma^1 + \Gamma^1_{00,1}W^0W^1\sigma^0 & \text{if } x^{(i)}x^{(i)} \leq R^2 \\ -\Gamma^1_{00,1}W^0W^0\sigma^1 + \Gamma^1_{00,1}W^0W^1\sigma^0 - \Gamma^1_{00,2}W^0W^0\sigma^2 + \Gamma^1_{00,2}W^0W^2\sigma^0 & \text{if } x^{(i)}x^{(i)} \geq R^2 \end{cases}$$

and

$$T^2 = \begin{cases} -\Gamma^2_{00,2}W^0W^0\sigma^2 + \Gamma^2_{00,2}W^0W^2\sigma^0 & \text{if } x^{(i)}x^{(i)} \leq R^2 \\ -\Gamma^2_{00,2}W^0W^0\sigma^2 + \Gamma^2_{00,2}W^0W^2\sigma^0 - \Gamma^2_{00,1}W^0W^0\sigma^1 + \Gamma^1_{00,2}W^0W^2\sigma^0 & \text{if } x^{(i)}x^{(i)} \geq R^2 \end{cases}$$

where $,i := \frac{\partial}{\partial x^{(i)}}$, and, remembering we have set $\lambda = T$, $W^0 = 1$. If we set $\sigma^0 = 0$, i.e. consider the neighboring autoparallels at the same time, and, for simplicity, consider the points in the manifold with $x^{(2)} = 0$, we have

$$T^1 = \begin{cases} -\frac{GM\sigma^1}{c^2R^3} & \text{if } x^{(i)}x^{(i)} \leq R^2 \\ \frac{2GM\sigma^1}{c^2(x^{(1)})^3} & \text{if } x^{(i)}x^{(i)} \geq R^2 \end{cases}$$

and

$$T^2 = \begin{cases} -\frac{GM\sigma^2}{c^2R^3} & \text{if } x^{(i)}x^{(i)} \leq R^2 \\ -\frac{GM\sigma^2}{c^2(x^{(1)})^3} & \text{if } x^{(i)}x^{(i)} \geq R^2 \end{cases}$$

or, w.r.t. ordinary time t and thereby also tidal force per unit mass,

$$\frac{F^1_{tidal}}{m} = \begin{cases} -\frac{GM\sigma^1}{R^3} & \text{if } x^{(i)}x^{(i)} \leq R^2 \\ \frac{2GM\sigma^1}{(x^{(1)})^3} & \text{if } x^{(i)}x^{(i)} \geq R^2 \end{cases}$$

and

$$\frac{F^2_{tidal}}{m} = \begin{cases} -\frac{GM\sigma^2}{R^3} & \text{if } x^{(i)}x^{(i)} \leq R^2 \\ -\frac{GM\sigma^2}{(x^{(1)})^3} & \text{if } x^{(i)}x^{(i)} \geq R^2 \end{cases}$$

The former describes the radial tidal forces, and the latter the azimuthal tidal forces, both of which must be observed in the local neighborhood of a freely-falling frame.

Moreover, an observer in a freely-falling frame, if defined globally, must, at any given instant, observe a 3-acceleration of the center of mass of the source matter, i.e. Earth, equal to that of a test-particle at the same point of the manifold in a globally defined frame with zero 3-acceleration w.r.t. the Earth, and in general we expect that the relative acceleration of test-particles w.r.t. the center of mass of the source matter is invariant under transformations into any accelerated reference frame. This defines the closest analogy to an inertial frame possible in a non-flat spacetime. See Figures 11 and 12.

4 Magnetic-type Inertiogravitational Field of Earth

Now we turn to evaluate $\nabla^2 \vec{A} = \frac{4\pi G\rho\vec{v}}{c^3}$. Locally, $\vec{v} = \vec{\omega}\times\vec{r}$. The general solution for $\vec{A}$ is now given by

$$\vec{A} = -\frac{G\rho}{c^3}\int_{volume}\frac{\vec{\omega}\times\vec{r'}}{|\vec{r}-\vec{r'}|}d\tau' + \vec{f}(x^i) \tag{15}$$

where $\vec{f}(x^i) = k_{ij}x^i$ is a linear vector function of the Cartesian coordinates. This integral is non-trivial, and we shall solve it following a method outlined in Griffiths, with modifications as needed. We first start by picking some point $\vec{r}$ at an angle θ w.r.t. $\vec{\omega}$, and then, for this point, considering a set of Cartesian axes such that $\vec{\omega}$ lies in the x-z plane. We then evaluate $\vec{\omega}\times\vec{r'}$ in terms of this set of axes, but $|\vec{r}-\vec{r'}|$ and $d\tau'$ in terms of spherical-polar coordinates w.r.t. this Cartesian set of axes, and then integrate this expression in $\vec{r'}$ over the volume of the sphere. We then reference the result to the global 3-coordinate system in which we have situated the rotating sphere.

In terms of the Cartesian axes,

$$\vec{\omega}\times\vec{r'} = r'\omega[-\cos\theta\sin\theta'\sin\phi'\hat{x}+(\cos\theta\sin\theta'\cos\phi'-\sin\theta\cos\theta')\hat{y}+\sin\theta sin\theta'\sin\phi'\hat{z}]$$

.

Since $d\tau' = dr'r'd\theta'd\phi'$, all terms contributing to $\vec{A}$ besides that coming from $-\sin\theta\cos\theta'\hat{y}$ are some constant multiplied by either $\int_0^{2\pi}\sin\phi' d\phi'$ or $\int_0^{2\pi}\cos\phi' d\phi'$, both of which evaluate to 0. Thus we have

$$\vec{A}(\vec{r}) = -\frac{G\rho\omega}{c^3}\int_0^R\int_0^\pi\int_0^{2\pi}\frac{-r'\sin\theta\cos\theta' r'^2\sin\theta' d\phi' d\theta' dr'}{\sqrt{r'^2+r^2-2rr'\cos\theta'}}\hat{y}$$

where we have used the law of cosines to find $|\vec{r}-\vec{r'}|$ and dropped the $\vec{f}(x^i)$ for the time being.

After evaluating the ϕ' and θ' limits, we have

$$\vec{A}(\vec{r}) = \frac{2\pi G\rho\omega\sin\theta}{c^3}\int_0^R r'^3 f(r')dr'\hat{y}$$

where

$$f(r') = \begin{cases} \frac{2r'}{3r^2} & \text{if } r' \leq r \\ \frac{2r}{3r'^2} & \text{if } r' \geq r \end{cases}$$

This gives

$$\vec{A}(r,\theta,\phi) = \frac{1}{c^3}\begin{cases} \frac{2\pi G\rho\omega\sin\theta(5R^2r-3r^3)}{15}\hat{y} & \text{if } r \le R \\ \frac{4\pi GR^5\rho\omega\sin\theta}{15r^2}\hat{y} & \text{if } r \ge R \end{cases}$$

By inspection, it is clear that our locally defined Cartesian coordinate system is related to the global, spherical polar coordinate system in which we have situated this rotating sphere in such a way that for every choice of $\vec{r}$, $\hat{\phi} = -\hat{y}$.

Thus, we finally have

$$\vec{A}(r,\theta,\phi) = \frac{1}{c^3}\begin{cases} -\frac{2\pi G\rho\omega\sin\theta(5R^2r-3r^3)}{15}\hat{\phi} & \text{if } r \le R \\ -\frac{4\pi GR^5\rho\omega\sin\theta}{15r^2}\hat{\phi} & \text{if } r \ge R \end{cases}$$

or, if we convert to cylindrical coordinates (r,ϕ,z) w/ the $z=0$ plane the equatorial plane,

$$\vec{A}(r,\phi,z) = \frac{1}{c^3}\begin{cases} -\frac{GM\omega r[5R^2-3(r^2+z^2)]}{10R^3}\hat{\phi} & \text{if } r^2+z^2 \le R^2 \\ -\frac{GMR^2\omega r}{5(r^2+z^2)^{\frac{3}{2}}}\hat{\phi} & \text{if } r^2+z^2 \ge R^2 \end{cases} \tag{16}$$

Rather than figure out what functional forms may be added to the above solution for $\vec{A}$ that satisfy the Laplacian in cylindrical coordinates, we consider equations (6) and (15) directly and consider how the case where we permit all $k_{ij}$to be non-zero, which we shall call the primed case, compares to the case in which we require all k_{ij} to be zero, thereby forcing $\vec{A}$ to be given by equation (16). It trivially follows that

$$\Gamma'^{(j)}_{(0)(i)} = \Gamma^{(j)}_{(0)(i)} + (k_{ij} - k_{ji})$$

.

We see in this simple Cartesian form that the freedom of adding functions f_n in the generalized coordinates to the components of $\vec{A}$, A_n, which satisy $\nabla^2 f_n(q^i) = 0$ corresponds to a freedom to choose, at any given point of the manifold, a rotating coordinate system. In order to first analyze the simplest case, we therefore choose to examine the $\Gamma^{(j)}_{(0)(i)}$ in a non-rotating frame, for which we use equations (6) and (16).

It follows that

$$\Gamma^{(\phi)}_{(t)(r)} = -\Gamma^{(r)}_{(t)(\phi)} = \frac{1}{c^2}\begin{cases} \frac{GM\omega(5R^2-3z^2-9r^2)}{10R^3} & \text{if } r^2+z^2 \le R^2 \\ \frac{GMR^2\omega(z^2-2r^2)}{5(r^2+z^2)^{\frac{5}{2}}} & \text{if } r^2+z^2 \ge R^2 \end{cases} \tag{17}$$

$$\Gamma^{(z)}_{(t)(\phi)} = -\Gamma^{(\phi)}_{(t)(z)} = \frac{1}{c^2}\begin{cases} \frac{3GM\omega rz}{5R^3} & \text{if } r^2+z^2 \le R^2 \\ \frac{3GMR^2\omega rz}{5(r^2+z^2)^{\frac{5}{2}}} & \text{if } r^2+z^2 \ge R^2 \end{cases} \tag{18}$$

$$\Gamma^{(z)}_{(t)(r)} = -\Gamma^{(r)}_{(t)(z)} = 0 \tag{19}$$

If we assume the spacetime to be holonomic, then it follows from equation (4) that $\Gamma^{(j)}_{(i)(0)} = \Gamma^{(j)}_{(0)(i)}$ for r, z, ϕ.

The $\Gamma^{(j)}_{(0)(i)}$, then, represent a rotation of the spatial basis vectors. By visual inspection, it is clear that positive $\Gamma^{(\phi)}_{(t)(r)}$ represents local rotation about the positive z axis, by the traditional right-hand-rule convention, while positive $\Gamma^{(z)}_{(t)(\phi)}$ represents local rotation about the positive r axis. Since infinitesimal rotations commute, we make take the net rotation rate from the $\Gamma^{(z)}_{(t)(\phi)}$ and $\Gamma^{(\phi)}_{(t)(r)}$ as a rotation rate about an axis in the local r-z plane, with the magnitude and direction of the rotation rate being determined by treating $\Gamma^{(z)}_{(t)(\phi)}$ and $\Gamma^{(\phi)}_{(t)(r)}$ as r and z components, respectively, of the angular velocity vector describing the local rotation of the space. Figure 13 shows precisely this for the Earth, with the shade of the arrows representing the relative rate of rotation.

The equations of motion for such a frame,

$$\frac{D^2x^{(r)}}{dt^2} = -\Gamma^{(r)}_{(t)(t)} - \Gamma^{(r)}_{(t)(\phi)}w^{(\phi)} - \Gamma^{(r)}_{(\phi)(t)}w^{(\phi)}$$

$$\frac{D^2x^{(\phi)}}{dt^2} = -\Gamma^{(\phi)}_{(t)(r)}w^{(r)} - \Gamma^{(\phi)}_{(t)(z)}w^{(z)} - \Gamma^{(\phi)}_{(z)(t)}w^{(z)}$$

$$\frac{D^2x^{(z)}}{dt^2} = -\Gamma^{(z)}_{(t)(t)} - \Gamma^{(z)}_{(t)(\phi)}w^{(\phi)} - \Gamma^{(z)}_{(\phi)(t)}w^{(\phi)}$$

, show that if we regard Figure 12 as representing a magnetic-type field, then, in this frame, we should expect to see what could be described as a magnetic-type force of the form $\vec{F} = m\vec{v} \times \vec{B}_{gravitational}$. Since we have $\Gamma^{(j)}_{(i)(0)} = \Gamma^{(j)}_{(0)(i)}$, the effective magnetic-type gravitational field is given by twice the magnitude of the $\Gamma^{(j)}_{(0)(i)}$ given by equations (17),(18), and (19). Qualitatively by visual inspection, we should expect that any particle freely falling should always track west ($(-\hat{r}_{spherical}) \times \vec{B}_{gravitational}$ is always in the $-\hat{\phi}$). This motion can be most properly understood geometrically and conceptually as the moving matter resulting in the local dragging or twisting of the inertial frames, i.e. the spacetime, and the moving particle merely tracing the local orientation of the space in the chronologically prior hypersurface of the spacetime, just as a moving particle in an absolutely rotating frame in a flat, holonomic spacetime merely traces the chronologically prior local orientation of a spatial hypersurface of the 4-d spacetime as described by the Coriolis force. In the above case of equatorial free-fall in a non-rotating frame, it may be conceptually helpful to think of the particle as tracking towards where the point on the surface of the Earth towards which it is falling was a moment prior, although it should be remembered that the rotating matter causes the rotation of the spatial basis vectors everywhere in the spacetime, as given by equations (17), (18), and (19), and not merely in the region of the spacetime in which the matter is rotating, and further that the azimuthal motion will then also result in a small force directed radially outward. A four-force-free observer at any point in the spacetime would observe their local frame of reference rotating by the amounts given by equations (17), (18), and (19) with respect to the background of stars.

Now, we interpret physically c as the speed of light, assigning it a numerical value of $c = 2.99 \times 10^8$m/s, in order to use equations (17), (18), and (19) as a near-field, low velocity description of the frame dragging of general relativity. In general and special relativity, the speed of light, or more generally, the fundamental velocity of special relativity, is by construction the fundamental conversion factor between space and time, tying together all reference frames with this constant of nature. Thus, if we would like to use the theory presented here as a limit of general relativity, we must use this numerical value. Using this value, we evaluate equation (17) at $r = R$ and $z = 0$, i.e. the equator, and find a value of $\Gamma^{(\phi)}_{(t)(r)} = -126.4$ mas / yr. To evaluate the effective magnetic-type gravitational field, we must multiply by 2 to account for the equal $\Gamma^{(\phi)}_{(r)(t)}$. In terms of s^{-1} and thereby also acceleration per velocity, this is $\vec{B}_{gravitational} = -3.895 \times 10^{-14} s^{-1}$. Thus it is unlikely that the magnetic-type forces acting on freely falling test-particles will themselves be detectable directly in the case of Earth.

It certainly may be objected that using a numerical value for the speed of light as our spacetime conversion factor incorporates insights from the special theory, when the expressed intent of this paper was to logically separate special relativity from the gravitational equivalence principle. It is true that we now know that the special theory must be incorporated into any general-relativistic view of spacetime, and the speed of light is the appropriate physical interpretation. This, however, does not invalidate the consideration that the gravitational equivalence principle being logically independent of any insights of the special theory of relativity allows us to formulate a non-flat affine Newtonian spacetime, which incorporates some conversion factor c to make commensurate units, and then allows one conceptually to formulate the invariant inertial tendencies of matter in terms of the only curvature tensor expressible in terms of the stress-energy tensor. Thus, this model of gravitation, including the functional dependencies of the $\Gamma^{(m)}_{(0)(0)}$ and $\Gamma^{(m)}_{(0)(n)}$ contained therein, serves as a *valid* starting point for the general theory in the near-field, low-velocity regime, even if we turn to the general theory itself to interpret this conversion factor physically and assign it a numerical value. History leaves us only to speculate how a Newtonian thinker equipped with the mathematical tool of the affine connection would actually interpret the conversion factor c, given that the idea of the affine connection was developed after the conception of the special theory of relativity, and thus was unavailable to Newtonian chronometers.

We now turn to analyze eqautions (17), (18), and (19) in a frame of reference which is rotating at constant Ω relative to the non-rotating frame in which the sphere is rotating at ω, about the axis of rotation of the sphere itself. We can relate this rotating frame of reference to a Cartesian spatial basis by setting the axis of rotation of the sphere and the rotating frame as $e^{\nu}_{(3)}$. Then, the Cartesian spatial basis vectors of the rotating frame are related to the non-rotating frame discussed above in the same way as the rotating frame of reference discussed in the introduction was related to the globally flat affine spacetime:

$$e'^{\nu}_{(0)} = e^{\nu}_{(0)} + \frac{\vec{\Omega}}{c} \times \vec{r} = e^{\nu}_{(0)} + \frac{\Omega x^{(1)}}{c} e^{\nu}_{(2)} - \frac{\Omega x^{(2)}}{c} e^{\nu}_{(1)}$$

, where $\vec{r} = x^{(m)} e^{\nu}_{(m)}$ and $e^{\nu}_{(m)} = \delta^{\nu}_{(m)}$,

$$e'^{\nu}_{(1)} = cos\frac{\Omega T}{c} e^{\nu}_{(1)} + sin\frac{\Omega T}{c} e^{\nu}_{(2)}$$

$$e'^{\nu}_{(2)} = -sin\frac{\Omega T}{c} e^{\nu}_{(1)} + cos\frac{\Omega T}{c} e^{\nu}_{(2)}$$

$$e'^{\nu}_{(3)} = e^{\nu}_{(3)}$$

$$e_\nu^{'(1)} = cos\frac{\Omega T}{c}e_\nu^{(1)} + sin\frac{\Omega T}{c}e_\nu^{(2)}$$
$$e_\nu^{'(2)} = -sin\frac{\Omega T}{c}e_\nu^{(1)} + cos\frac{\Omega T}{c}e_\nu^{(2)}$$
$$e_\nu^{'(3)} = e_\nu^{(3)}$$

It follows that

$$\Gamma^{'(1)}_{(0)(0)} = -\frac{\Omega^2}{c^2}(x^{(1)}cos\frac{\Omega T}{c}+x^{(2)}sin\frac{\Omega T}{c})+\frac{\Omega}{c}\big(x^{(1)}cos\frac{\Omega T}{c}(\Gamma^{(1)}_{(0)(2)}+\Gamma^{(1)}_{(2)(0)})-x^{(2)}sin\frac{\Omega T}{c}(\Gamma^{(2)}_{(0)(1)}+\Gamma^{(2)}_{(1)(0)})\big)...$$
$$+cos\frac{\Omega T}{c}\Gamma^{(1)}_{(0)(0)} + sin\frac{\Omega T}{c}\Gamma^{(2)}_{(0)(0)} \rightarrow$$

$$\Gamma^{'(1)}_{(0)(0)} = \big(-\frac{\Omega^2}{c^2}+\frac{\Omega}{c}(\Gamma^{(1)}_{(0)(2)}+\Gamma^{(1)}_{(2)(0)})\big)x^{'(1)}+cos\frac{\Omega T}{c}\Gamma^{(1)}_{(0)(0)}+sin\frac{\Omega T}{c}\Gamma^{(2)}_{(0)(0)} \quad (20)$$

,

$$\Gamma^{'(2)}_{(0)(0)} = -\frac{\Omega^2}{c^2}(-x^{(1)}sin\frac{\Omega T}{c}+x^{(2)}cos\frac{\Omega T}{c})-\frac{\Omega}{c}\big(-x^{(1)}sin\frac{\Omega T}{c}(\Gamma^{(1)}_{(0)(2)}+\Gamma^{(1)}_{(2)(0)})-x^{(2)}cos\frac{\Omega T}{c}(\Gamma^{(2)}_{(0)(1)}+\Gamma^{(2)}_{(0)(1)})\big)...$$
$$-sin\frac{\Omega T}{c}\Gamma^{(1)}_{(0)(0)} + cos\frac{\Omega T}{c}\Gamma^{(2)}_{(0)(0)} \rightarrow$$

$$\Gamma^{'(2)}_{(0)(0)} = \big(-\frac{\Omega^2}{c^2}-\frac{\Omega}{c}(\Gamma^{(2)}_{(0)(1)}+\Gamma^{(2)}_{(1)(0)})\big)x^{'(1)}-sin\frac{\Omega T}{c}\Gamma^{(1)}_{(0)(0)}+cos\frac{\Omega T}{c}\Gamma^{(2)}_{(0)(0)} \quad (21)$$

.

Noting that $\Gamma^{'(r)}_{(0)(0)}e^{'\mu}_{(r)} = \Gamma^{'(1)}_{(0)(0)}e^{'\mu}_{(1)} + \Gamma^{'(2)}_{(0)(0)}e^{'\mu}_{(2)}$, where r is once again the cylindrical-polar radial coordinate/basis vector, using equations (12) and (13), we have

$$\Gamma^{'(r)}_{(0)(0)}e^{'\mu}_{(r)} = \big(-\frac{\Omega^2}{c^2}+\frac{\Omega}{c}(\Gamma^{(1)}_{(0)(2)}+\Gamma^{(1)}_{(2)(0)})\big)(x^{'(1)}e^{'\mu}_{(1)}+x^{'(2)}e^{'\mu}_{(2)}) + ...$$
$$\Gamma^{(1)}_{(0)(0)}(e^{'\mu}_{(1)}cos\frac{\Omega T}{c} - e^{'\mu}_{(2)}sin\frac{\Omega T}{c}) + \Gamma^{(2)}_{(0)(0)}(e^{'\mu}_{(1)}sin\frac{\Omega T}{c} + e^{'\mu}_{(2)}cos\frac{\Omega T}{c}) \rightarrow$$

$$\Gamma^{'(r)}_{(0)(0)}e^{'\mu}_{(r)} = \big(-\frac{\Omega^2}{c^2}+\frac{\Omega}{c}(\Gamma^{(1)}_{(0)(2)}+\Gamma^{(1)}_{(2)(0)})\big)(x^{'(1)}e^{'\mu}_{(1)}+x^{'(2)}e^{'\mu}_{(2)}) + ...$$
$$\Gamma^{(1)}_{(0)(0)}e^{\mu}_{(1)} + \Gamma^{(2)}_{(0)(0)}e^{\mu}_{(2)} \rightarrow$$

$$\Gamma^{'(r)}_{(0)(0)}e^{'\mu}_{(r)} = \big[\big(-\frac{\Omega^2}{c^2}+\frac{\Omega}{c}(\Gamma^{(1)}_{(0)(2)}+\Gamma^{(1)}_{(2)(0)})\big)r + \Gamma^{(r)}_{(0)(0)}\big]e^{'\mu}_{(r)} \rightarrow$$

$$\Gamma^{'(r)}_{(t)(t)} = -\Omega^2 r + \Omega r(\Gamma^{(r)}_{(t)(\phi)} + \Gamma^{(r)}_{(\phi)(t)}) + \Gamma^{(r)}_{(t)(t)} \quad (22)$$

The first term is the orginary centrifugal force, the second the magnetic-type interaction from being in the rotated frame of reference, giving $v^{(\phi)} = \Omega r$, and the third the electric-type interaction from the non-rotating frame of reference.

It is clear that because $\Gamma^{(\phi)}_{(t)(t)} = 0$, being in a rotating reference frame does not by itself imply radial velocity (hence no azimuthal magnetic-type gravitational deflection from this alone), and the centrifugal force is purely radial, that $\Gamma^{'(\phi)}_{(t)(t)} = 0$.

We now evaluate the Coriolis-type t.c.c. in this rotating frame of reference:

$$\Gamma^{'(1)}_{(0)(2)} = \Gamma^{'(1)}_{(2)(0)} = \Gamma^{(1)}_{(0)(2)} - \frac{\Omega}{c} \rightarrow$$
$$\Gamma^{'(r)}_{(t)(\phi)} = \Gamma^{'(r)}_{(\phi)(t)} = \Gamma^{(r)}_{(t)(\phi)} - \Omega$$

,

$$\Gamma^{'(2)}_{(0)(1)} = \Gamma^{'(2)}_{(1)(0)} = \Gamma^{(2)}_{(0)(1)} + \frac{\Omega}{c} \rightarrow$$
$$\Gamma^{'(\phi)}_{(t)(r)} = \Gamma^{'(\phi)}_{(r)(t)} = \Gamma^{(\phi)}_{(t)(r)} + \Omega$$

.

We are justified in equating $\Gamma^{'(2)}_{(0)(1)}$ with $\Gamma^{'(\phi)}_{(0)(r)}$ and $\Gamma^{'(1)}_{(0)(2)}$ with $\Gamma^{'(r)}_{(0)(\phi)}$ because in each case, the two rotating spatial vectors have the same relative orientation in the manifold, and the two sets differ from each other only by a (spatially) postionally dependent phase.

We are justified in equating $\Gamma^{'(2)}_{(1)(0)}$ with $\Gamma^{'(\phi)}_{(r)(0)}$ and $\Gamma^{'(1)}_{(2)(0)}$ with $\Gamma^{'(r)}_{(\phi)(0)}$ by similar logic: $e^{'\mu}_{(1)}$ and $e^{'\mu}_{(2)}$ have the same relative orientation as $e^{'\mu}_{(r)}$ and $e^{'\mu}_{(\phi)}$, and $e^{'\mu}_{(1)}$ and $e^{'\mu}_{(2)}$ take on all possible directions in the (1)-(2) 2-hypersurface, since they are rotating w.r.t. T, just as do $e^{'\mu}_{(r)}$ and $e^{'\mu}_{(\phi)}$. Thus it must be that $\Gamma^{'(2)}_{(1)(0)} = \Gamma^{'(\phi)}_{(r)(0)}$ and $\Gamma^{'(1)}_{(2)(0)} = \Gamma^{'(r)}_{(\phi)(0)}$.

If we set $\Omega = \omega$, looking at equations (17) and (22), we can see that objects resting on the Earth between the latitudes of 54.7^o N and 54.7^o S (the latitudes at which $z^2 = 2r^2$) will have a decrease in the centrifugal force per unit mass on the order of $2\Gamma^{(\phi)}_{(t)(r)}|_{r=R,z=0}\Omega_E R_E = 5 \times 10^{-12}\ ms^{-2}$ and those above 54.7^o N or below 54.7^o S will have an increase in the centrifugal force on the order of $2\Gamma^{(\phi)}_{(t)(r)}|_{r=R,z=0}\Omega_E \frac{R_E}{2} = 2 \times 10^{-12}\ ms^{-2}$, while the centrifugal force per unit mass itself is on the order of $f*(\Omega_E)^2 R_E = f*0.068\ ms^{-2}$ where f is the fraction of Earth's radius the point on the surface is from the axis of rotation. Equations (17) and (22) can be used to evaluate the exact predicted modification of the local gravitational field at the surface of the surface from this simplified model of Earth, but, as we have shown, the magnetic-type modification is very small compared to the ordinary centrifugal force.

Equations (17) and (22) can also be used to predict a splitting of stable circular orbital radii in the equatorial plane of a rotating sphere depending on whether the orbit is in the same direction or counter to the direction of sphere's rotation, analogous to the Zeeman effect.

In the equatorial plane, Equations (17) and (22) give

$$\Gamma'^{(r)}_{(t)(t)} = -\Omega^2 r + \Omega r \frac{4GMR^2\omega}{5c^2r^3} + \frac{GM}{r^2} \rightarrow$$

$$\Gamma'^{(r)}_{(t)(t)} = -\Omega^2 r + \frac{GM}{r^2}(1 + \frac{4\Omega\omega R^2}{5c^2})$$

The radius of stable circular orbit is such that, for a given Ω, $\Gamma'^{(r)}_{(t)(t)} = 0$:

$$r = (\frac{GM}{\Omega^2})^{\frac{1}{3}}(1 + \frac{4\Omega\omega R^2}{5c^2})^{\frac{1}{3}}$$

$(\frac{GM}{\Omega^2})^{\frac{1}{3}}$ is the stable orbital radius neglecting magnetic effects, which we will now call r_0.

If we call the dimensionless $\frac{4\Omega\omega R^2}{5c^2} = u$ a dummy variable, we can Taylor expand our expression for r, giving

$$r \approx r_0 \frac{1}{3}(1+u)^{-\frac{2}{3}}|_{u=0} u = \frac{r_0 * u}{3}$$

Since the splitting of stable circular orbital radii occurs when we consider $\Omega \rightarrow \pm\Omega$, we send $u \rightarrow \pm u$ and take Δr as $r(+u) - r(-u)$, giving

$$\Delta r \approx \frac{2}{3} u * r_0 \rightarrow$$

$$\Delta r \approx \frac{8\Omega\omega R^2}{15c^2} r_0$$

In the case of Earth, the magnitude of the splitting at the orbital radius of $r = 2R_E$ is

$$\Delta r = 9.84 \times 10^{-5} m = 98.4 \mu m$$

We should expect that non-circular and non-equatorial orbits are also modified by the frame-dragging effects.

5 Gravity Probe B

We now use the equations developed in the previous section –(17), (18), and (19) –to predict the amount of rotation Gravity Probe B should have undergone under just the considerations of this paper. Gravity Probe B (GP-B) was a gyroscope launched by NASA into an approximately circular orbit at an inclination of 90.007^o, i.e. orbiting from rotational pole to rotational pole, designed to measure the rotation the gyroscope would undergo due to the geodetic effect and due to the frame-dragging effect. In such an orbit, the symmetry of the frame-dragging effect means that the only net rotation the gyroscope would undergo would be due to the $\Gamma^{(\phi)}_{(t)(r)}$, given by equation (17). We can make an approximation for the amount of rotation the gyroscope should undergo by treating the orbit as perfectly circular, with radius r_o with an inclination of exactly 90^o. Equation (17) then becomes

$$\Gamma^{(\phi)}_{(t)(r)} = \frac{GMR^2\omega(z^2 - 2r^2)}{5r_o^5c^2}$$

Parameterized in terms of angle w.r.t. the axis of rotation θ, i.e. $z = r_o \cos\theta$ and $r = r_o \sin\theta$, this becomes

$$\Gamma^{(\phi)}_{(t)(r)}(\theta) = \frac{GMR^2\omega}{5r_o^3c^2}(\cos^2\theta - 2\sin^2\theta)$$

,

which takes on an average value over the orbit of

$$< \Gamma^{(\phi)}_{(t)(r)} >= -\frac{GMR^2\omega}{10r_o^3c^2}$$

According to NASA's Earth Fact Sheet, Earth has a mean volumetric radius of R = 6371 km and a total mass of M = 5.972 x 10^{24} kg. According to the National Space Science Data Center, GP-B had an apogee of h_a = 645 km and a perigee of h_p = 641 km (both as measured from the Earth's surface), and so can be ascribed a mean orbital radius to be used in our calculation of $r_o = R + \frac{h_a+h_p}{2} = 7014$ km. Taking Earth's angular velocity of rotation to be 2π radians divided into a sidereal day –23 hours, 56 minutes, and 4.091 seconds (5) –, this gives an average rotation of $< \Gamma^{(\phi)}_{(t)(r)} >= -3.024$ x 10^{-15} rad/s = −24.9 mas/year, i.e. 24.9 mas/year in the direction opposite Earth's rotation.

6 Concluding Remarks

We have seen that we can develop much of the conceptual and mathematical framework neccessary for arriving at a meaningful understanding of the general theory of relativity without ever introducing the concepts of relativity of simultaneity, time dilation, length contraction, or, more generally, a flat, locally Minkowskian spacetime, which offer their own unique learning challenges. This conceptual and mathematical framework is precisely what is neccessary and sufficient for understanding the gravitational equivalence principle and formulating equations of motion in a curved, although Newtonian, spacetime. Thus pedagogically, it is advantageous to develop these two sets of concepts separately – as they are much easier to understand in this way – and then consider the consequences of requiring that the postulates of special relativity apply to the tangent spaces of the manifold considered here.

This model of gravitation also serves as a valid near-field, low velocity limit and starting point for an approximation scheme of the general theory, owing to the logical independence of the gravitational equivalence principle and Lorentz invariance, although of course they must both be integrated into a general relativitistic description of spacetime. Indeed this is what allows us to interpret our spacetime conversion factor c as the speed of light in order to use this spacetime as a near-field approximation of the general theory of relativity. However – at the risk of being redundant and in the interest of stressing the conceptual importance of this point – these two considerations inform our view of the nature of space and time *independently* of one another.

Potential further work includes mathematically rigorously comparing the t.c.c. derivable from the Kerr metric to the non-metric t.c.c. presented here and formulating expressions for the tetrad components of the affine curvature tensor and of the equation of autoparallel deviation given the magnetic-type t.c.c.

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Figure 1: The same autoparallel curve has constant components under one coordinatization of the sphere (left) but non-constant components under a rotation of the original coordinatization of the sphere (right)

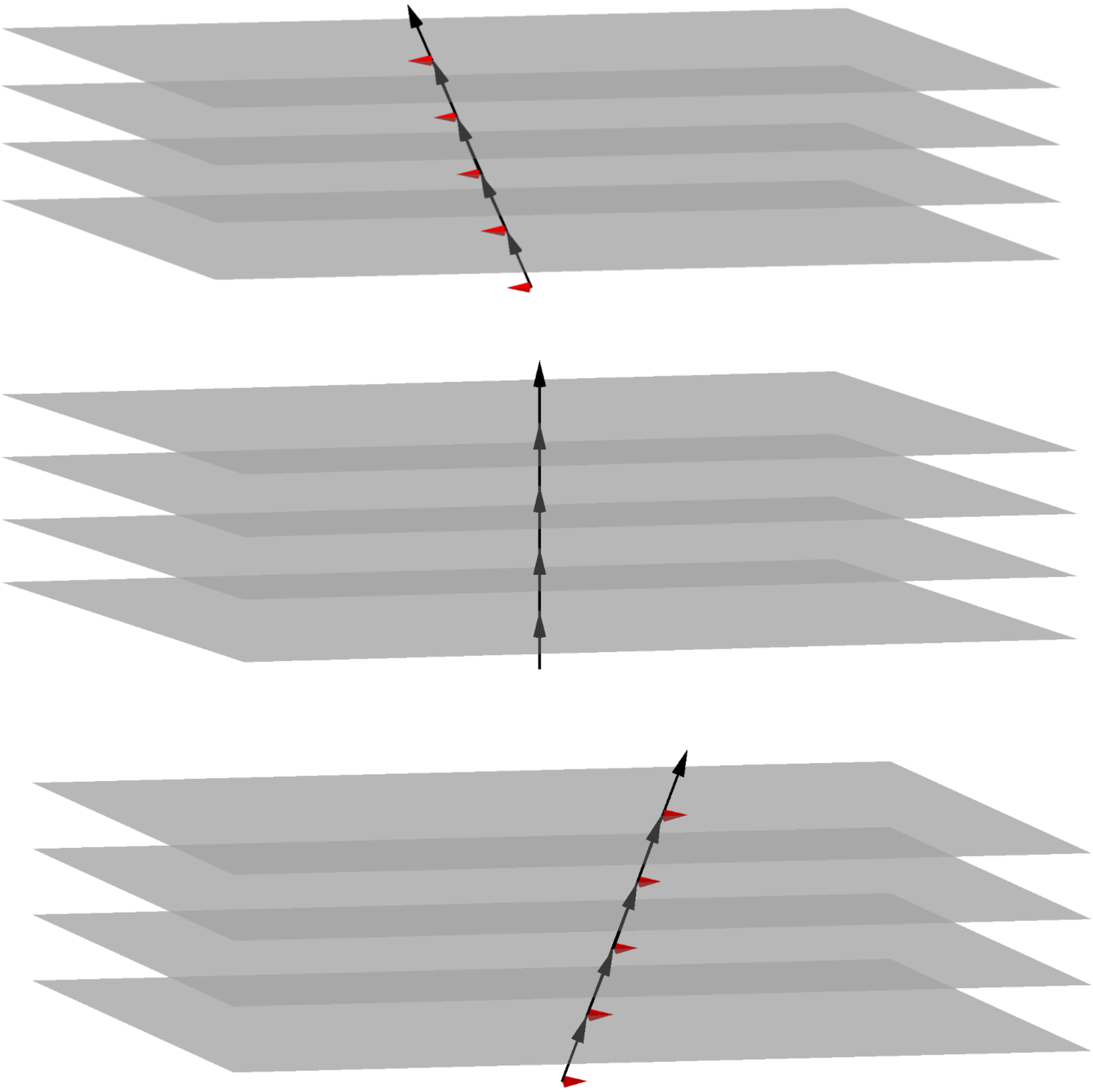

Figure 2: The above three diagrams show three possible inertial world-lines of force-free test particles and their 4-velocities which define autoparallels in a globally flat affine space. The surfaces represent spatial hypersurfaces T = const., compressed to two dimensions, and the red arrows are the spatial components of the 4-velocities. The three diagrams can also be thought of as representing the same inertial world-line and 4-velocities under an inertial change of coordinates

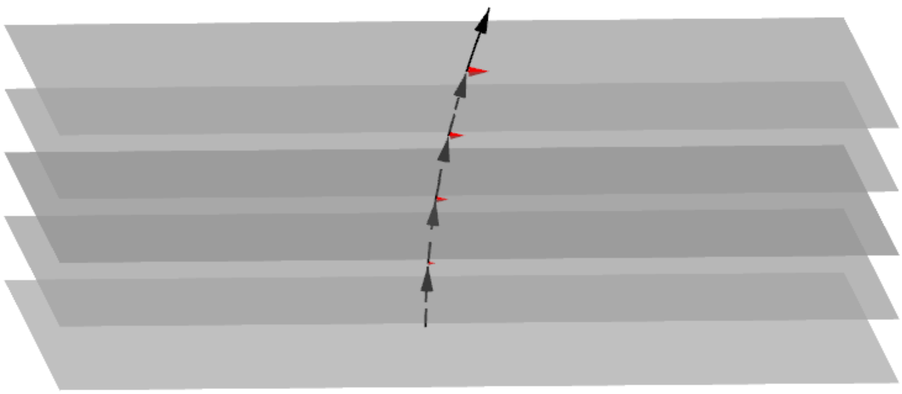

Figure 3: Newton's Second Law, $a = \frac{F}{m}$, states that a particle's worldline deviates from affine-flat autoparallel transport when subjected to a net force

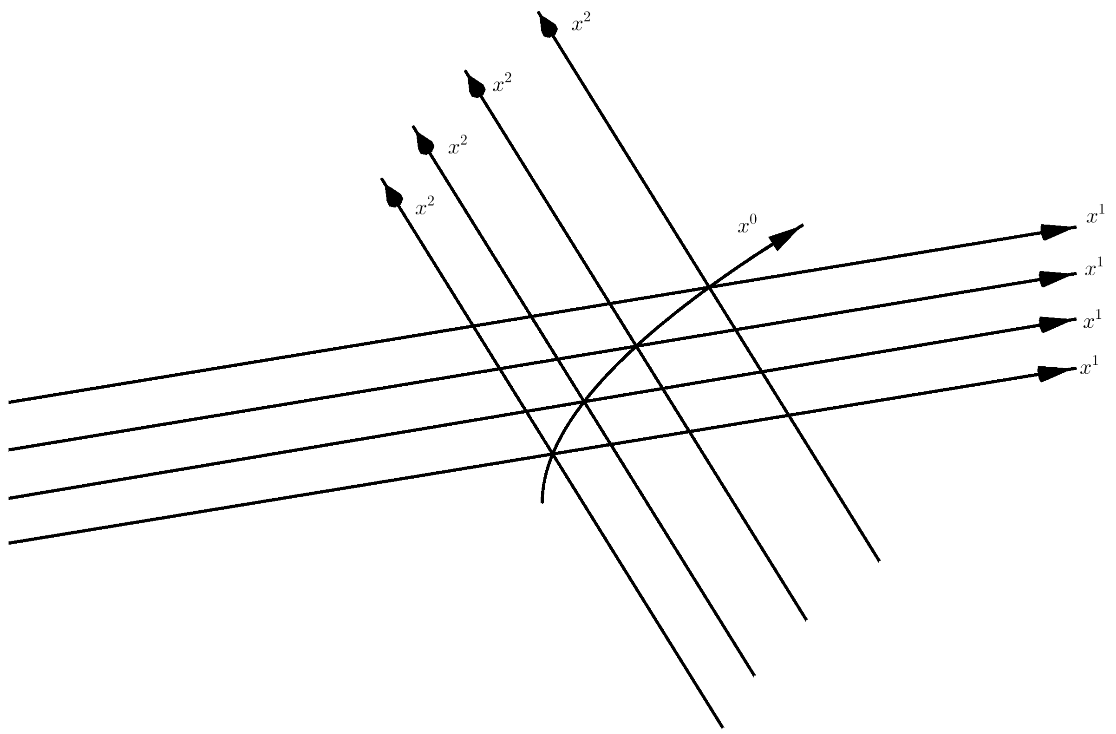

Figure 4: A coordinitization of a 4-space. The axis specifiy the $x^{(3)}$ coordinatization is not shown in order to visualize the temporal character of a given coordinatization. This coordinization can extend over the entirety of the manifold, or characterize the tangent space of a given point or set of points of the manifold

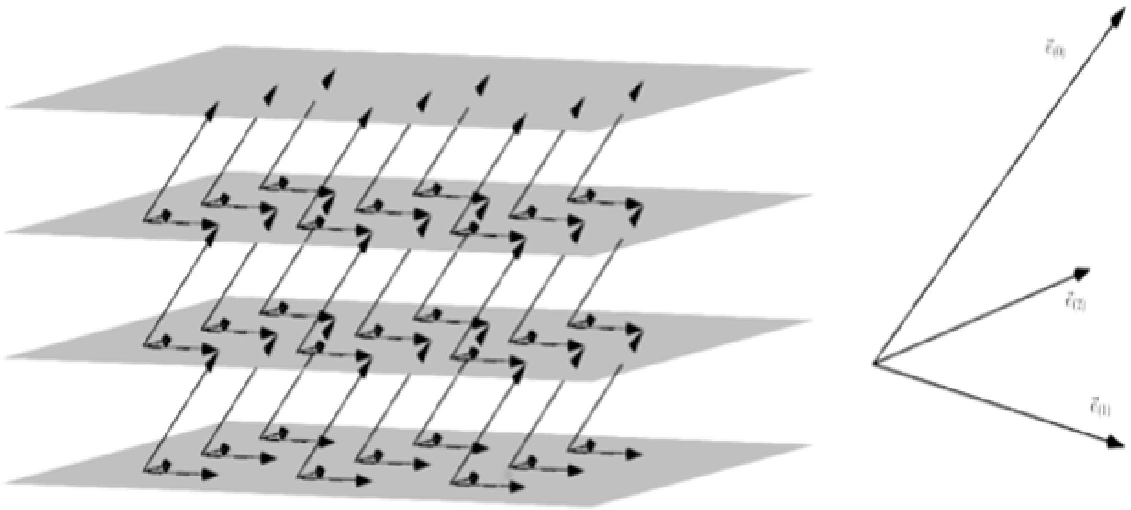

Figure 5: A specification of a tetrad field, without the $e^{\nu}_{(3)}$ field shown to visualize the temporal character of the frame of reference being defined by such a field. The surfaces represent spatial hypersurfaces compressed to two dimensions. Such a tetrad field can be specified over the entirety of the manifold, or over the tangent space of a given point or set of points of the manifold. While this particular tetrad field defines a uniformly translating frame of reference, the $e^{\nu}_{(0)}$ field can be non-constant, which would allow for accelerating or rotating reference frames, and the $e^{\nu}_{(i)}$ field can be non-constant, e.g. for a spherical polar spatial basis.

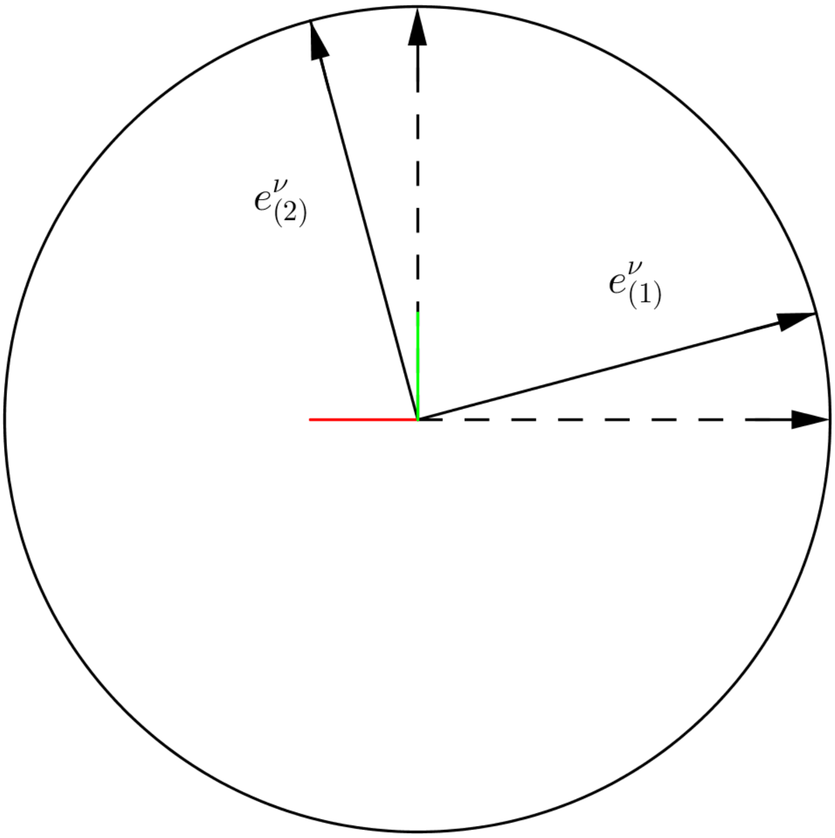

Figure 6: The green segment represents $\Gamma^{(2)}_{(0)(1)}$, while the red segment represents $\Gamma^{(1)}_{(0)(2)}$. Clearly $\Gamma^{(2)}_{(0)(1)} = -\Gamma^{(1)}_{(0)(2)}$ implies a rotation of $e^{\nu}_{(1)}$ and $e^{\nu}_{(2)}$.

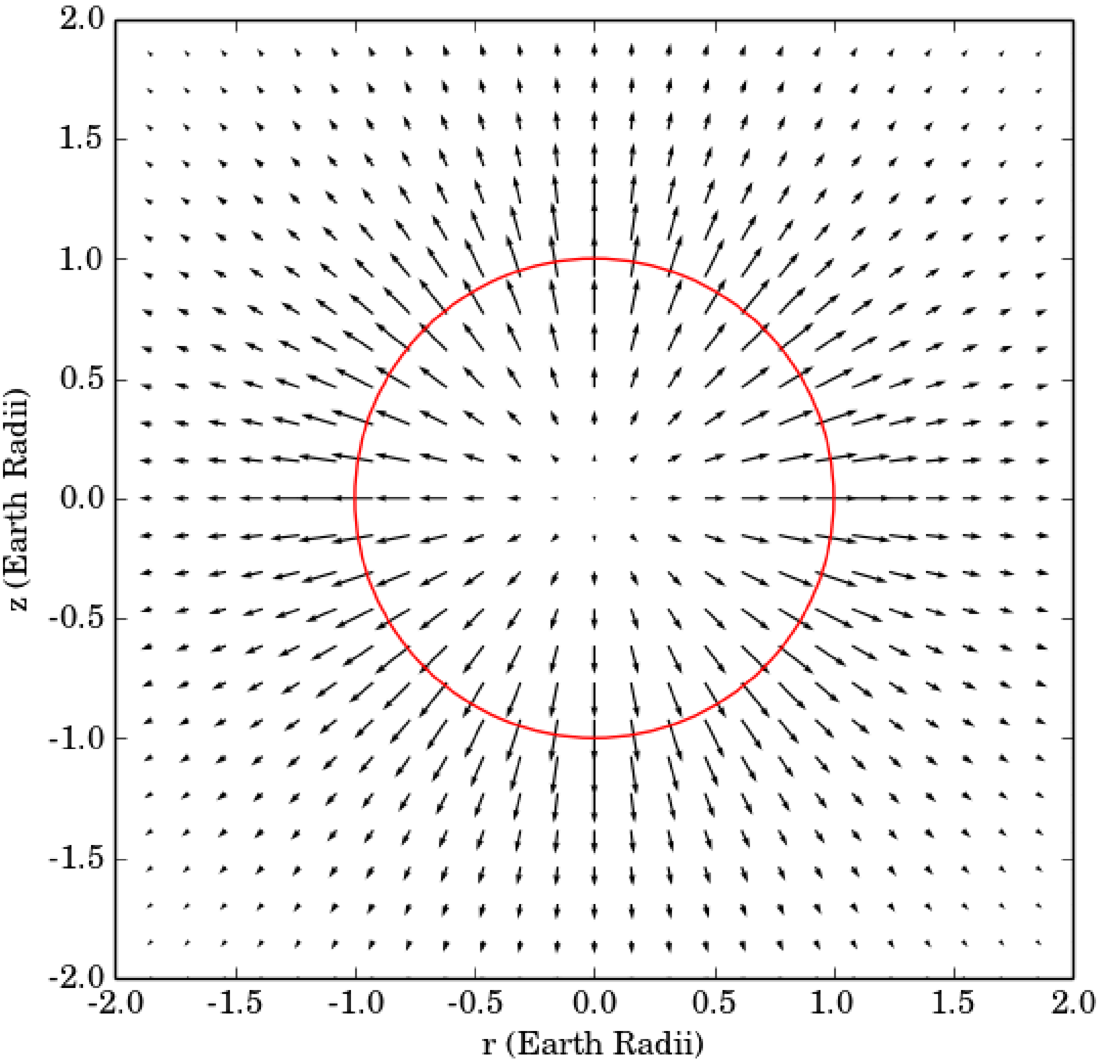

Figure 7: Cross-section of field showing local four-force per unit mass required to keep objects in a globally defined frame of reference with zero three-acceleration w.r.t. the sphere, i.e. with all $a_{(m)} = 0$

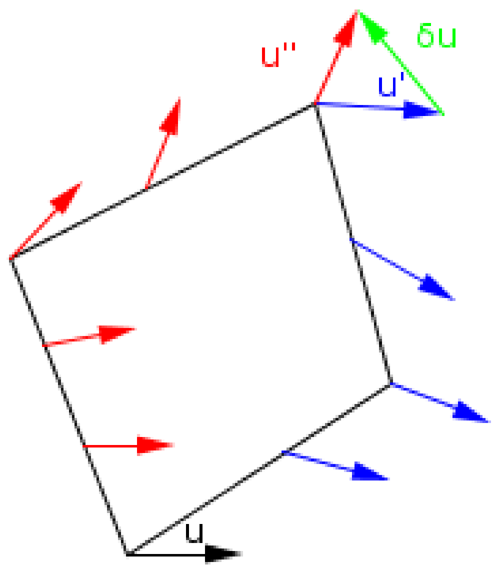

Figure 8: The path dependence of the parallel transport of a vector in a infinitesimal loop depends on the local value of the affine curvature tensor in the manifold

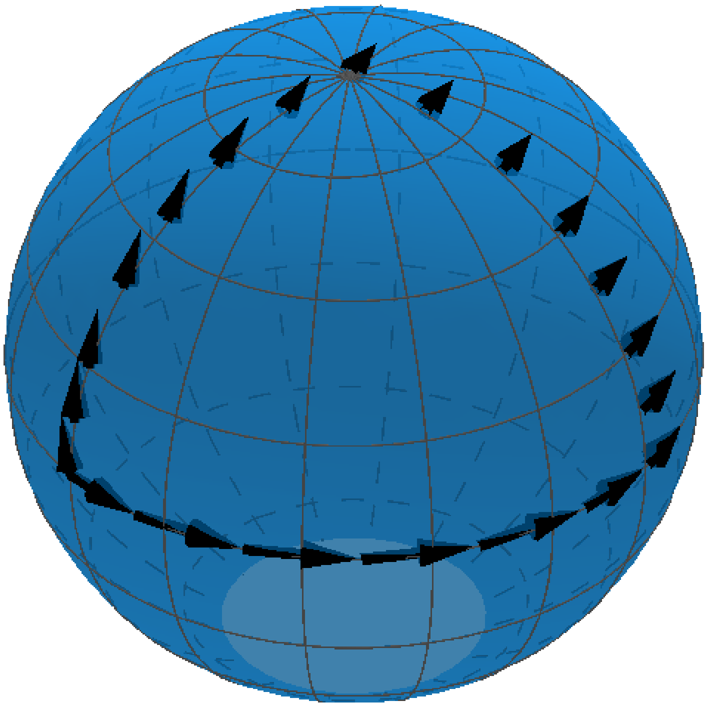

Figure 9: Visual demonstration of the change in a vector from parallel transporting it around a closed path on the surface of a sphere, showing the path dependence of the parallel transport of a vector in non-flat manifold

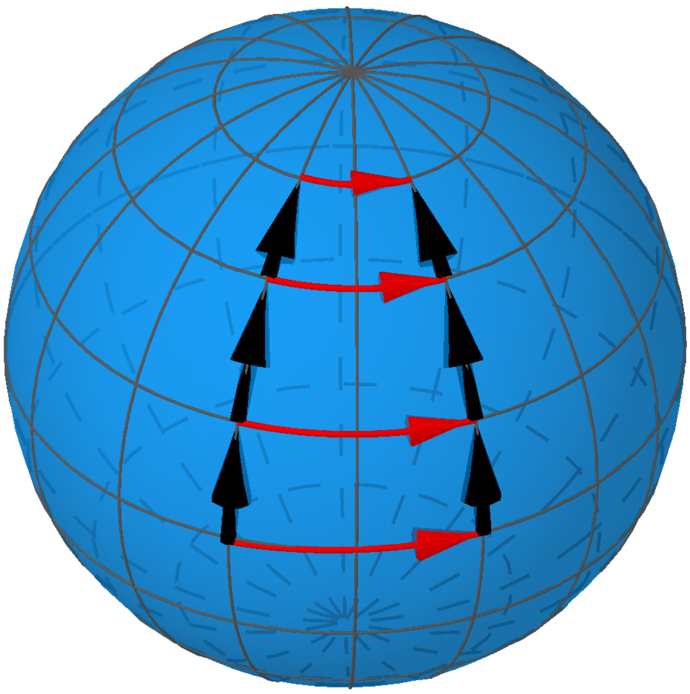

Figure 10: Evolution of a separation vector of two autoparallels on the surface of a sphere, which is independent of the coordinatization of the surface of the sphere

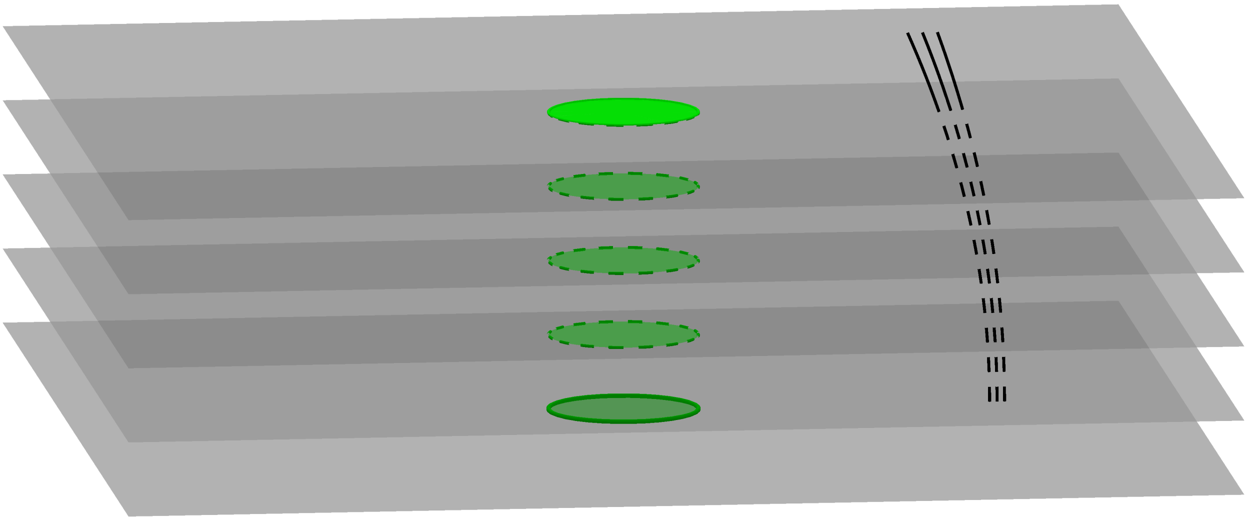

Figure 11: World-lines of three closely space four-force free test particles starting at rest in the spacetime modified by a uniform, rigid sphere, which has been compressed into the circle which is the intersection of the sphere with any plane containing the projection of the particles' worldlines onto the hypersurfaces, in a globally defined frame of reference with zero three-acceleration w.r.t. the center of mass of the sphere

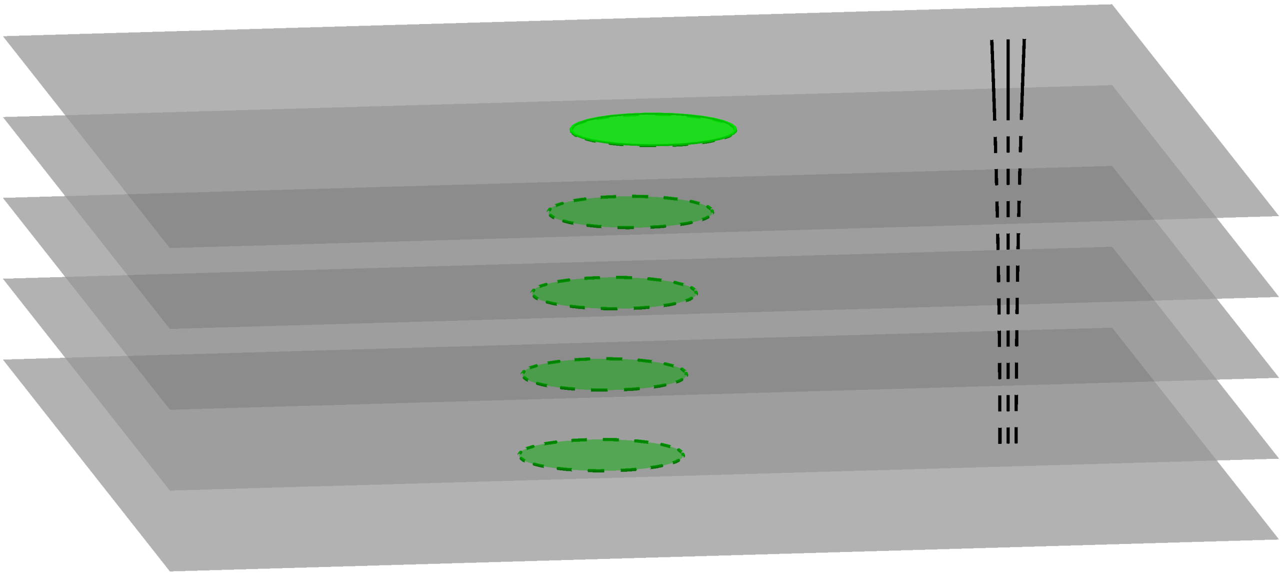

Figure 12: The worldlines of Figure 10 in a globally defined frame of reference with a constant three-acceleration throughout the manifold such that it is locally coaccelerating with the middle test particle. The center of mass of the source matter (the sphere) has the negative of this three-acceleration, and the time evolution of the separation vector of the autoparallels is the same as in Figure 10. This defines the closest analogy possible to a Newtonian inertial frame in a non-flat Newtonian spacetime

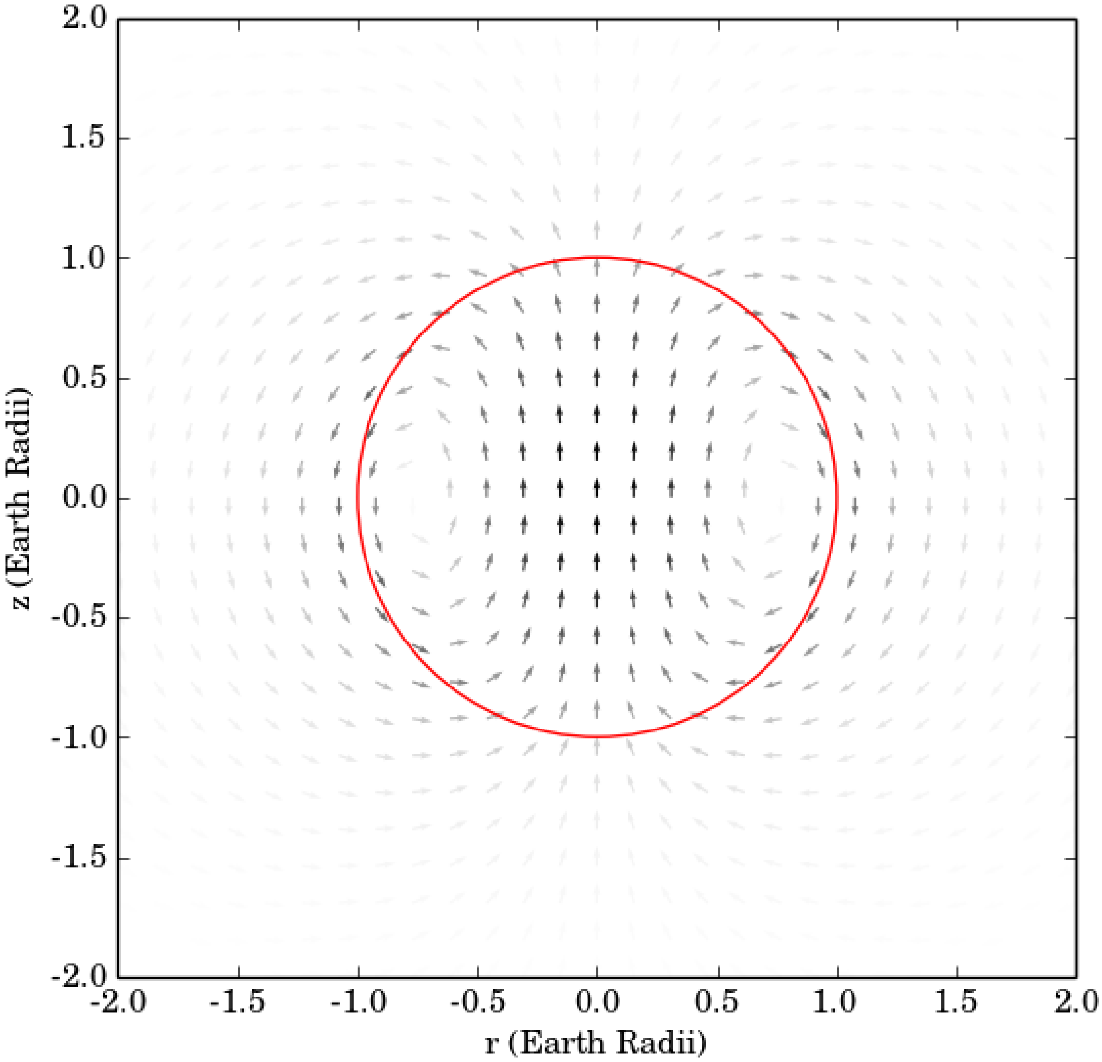

Figure 13: Cross-section of field representing local rotation of the space caused by the rotating source matter in a non-rotating frame, with the direction of the vector indicating the direction of the rotation by the traditional right-hand rule, and the shading of the vector indicating the magnitude of the rotation. The equations of motion tell us we may regard this figure as an effective gravitational “magnetic-field“